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A solution of a rectifier filter circuit with a capacitive input

D. L. Waidelich
Iowa State College

A SOLUTION OF A RECTIFIER FILTER CIRCUIT WITH A CAPACITIVE INPUT

by

D. L. Waidelich

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I. INTRODUCTION

A. The Full-Wave Rectifier Circuit with a Capacitive-Input Filter

Since about 1928 most radio receivers have taken all of the power needed to operate them from alternating voltage sources. The heaters of the tubes are supplied by alternating voltage stepped down through a transformer to the several voltages required. The plates of the tubes, on the other hand, require a direct voltage of the order of two or three hundred volts and are supplied by a transformer, rectifier, and filter circuit which steps up the alternating voltage, rectifies it to become pulsating direct current, and then filters it to produce constant direct current. During the last dozen years this type of direct-current power supply has been standard in almost every device employing electronic tubes and operating from alternating voltage sources. Among these devices might be included amplifiers, cathode-ray oscilloscopes, oscillators, television receivers, and radar equipment.

There are many variations of the rectifier and filter circuit that are used in these devices, but the most popular one probably is that of the full-wave rectifier circuit with a condenser-input filter circuit. This type of filter evolved quite naturally from the low-pass type of filter used in telephony work. At first it was sufficient to use the well-known attenuation formulas for a low-pass filter to design the smoothing filter necessary for a given amount of pulsation or ripple voltage at the power-supply load. It was very quickly noticed, however, that the filter had a profound influence on the operation of the rectifier, and the two could not with any degree of success be considered separately.

This realization led to analyses of the circuit which attempted to treat the circuit not as a filter problem, but as a circuit problem modified by the behavior of the tubes. Much of the difficulty encountered in analyzing the circuit stems from the non-linearity of the tube characteristics. One method of overcoming this difficulty that has met considerable success in practical applications of rectifier circuits is that of treating the tube-voltage-drop characteristic as made of two straight lines. Another difficulty is that of finding the steady-state solution of the problem without the use of Fourier series, since to approximate the impulsive type of current flowing in the tube requires too many terms in the finite Fourier series. A new steady-state operational calculus was developed to obviate this difficulty and was found to be useful not only in rectifier circuit studies but in all circuits studies involving repeated pulses and non-sinusoidal wave forms.

All of the analyses made so far are limited in their application to actual circuit design by the fact that the basic assumptions made are so severe that the analyses are useful only in a small region. In some cases the results of the analyses are useful only for one value of a parameter such as one frequency. An analysis of the rectifier circuit is needed that will liberalize the basic assumptions sufficiently so that the circuit operation can be presented over the whole range of circuit parameters involved and that is exact enough so that the results may be used in the design of the circuit. It is the purpose of this investigation to present such an analysis along with its results.

B. Review of Literature

In 1931, Wheatcroft[12] used two straight lines to represent the tube characteristic and applied a method of successive approximations to a Fourier series to obtain the current wave forms of the full-wave single-phase rectifier circuit with a simple inductive filter. He discussed the application of this method to a condenser-input filter circuit but did not actually use it. Lee[3] in 1932 showed how the half-wave condenser-input circuit could be solved approximately with the chief assumptions that there was no transformer, tube, or filter inductor voltage drop and that the filter inductance was infinite. The same year Terman[8] made an analysis of the circuit taking into account the tube and transformer drop and making the assumptions that the condenser charged instantaneously at the peak of the applied voltage wave and that the filter inductance was infinite. For the first time characteristic curves suitable for design work were given, although the assumption that the condenser charged instantaneously vitiated the results except for light loads. Freeman[1] in 1934, while a graduate student under Terman, made an improved analysis assuming that the tube had a constant resistance while conducting and the filter inductance was infinitely large. The results were presented in the form of design curves which were valid unfortunately only for one frequency and which covered a rather narrow range of values. The results of this thesis appeared in two other works by Terman[9, 10].

In 1935 Stout[7] used the method of succesive approximations to the Fourier series to solve the circuit using the assumption that the tube had zero voltage drop when conducting. One example was given with circuit values specified and with the circuit wave forms shown. No characteristic curves or design information were presented. Ludwig[4] in 1938 assumed that the input capacitance was infinitely large and presented characteristic curves taking into account the effect of tube resistance and transformer resistance and reactance. This analysis gave good results at light loads, but the results progressively became worse as the load increased. In 1943 Mitchell[6] made an analysis assuming that the transformer reactance was negligible, the tube had a constant resistance while conducting, that the filter inductance was infinitely large, and that the condenser charged linearly. Again this analysis seemed to give good results at light loads but not so good at heavier ones.

All of the analyses excepted those of Stout[7] and Freeman[1] involved assumptions that became less valid as the load became heavier. Only the analysis of Stout[7] could be used for finite filter inductances, but the necessity for including so many terms in the Fourier series probably is the reason why it has not been used. It is apparent that an analysis is needed that will be usable from open-circuit to short-circuit, that will describe the action of the circuit all of the way from very large filter inductances to ones so small that the phenomenon of filter resonance appears, and that will give characteristic curves that may be used in the design of rectifier and filter circuits.

II. THE STEADY-STATE OPERATIONAL CALCULUS

A. The Transforms

Originally it was decided to try to solve the equivalent circuit by the use of differential equations. While theoretically feasible, this method, in practice, entailed so many difficulties, particularly that of determining the arbitrary constants, that it was abandoned. The newly developed steady-state operational calculus[11] was next considered, because there would be no arbitrary constants to evaluate and there would be no transient terms mixed up with the desired steady-state solution.

Since this calculus has not been applied to any other problem as yet, the principles involved will be reviewed. The central idea of this operational calculus is the transform. Actually there are two transforms, a direct and an inverse transform. The direct transform operates on any time function such as a voltage or current and produces the voltage or current expressed in the operational form. The inverse transform reverses this procedure in that it operates on any operational form such as a transformed voltage or current and produces the original time function which is the actual voltage or current. The simplest way then of applying this calculus to a circuit is to obtain the direct transform of the applied voltage and divide the transformed voltage by the operational impedance of the circuit. The quotient of the transformed voltage divided by the operational impedance is the transformed current, and this transformed current may be operated upon by the inverse transform to produce the actual current, a time function. Essentially the above is an outline of the method used in the present case.

The direct transform is

$$S\left[f(t)\right] = \int_0^T \epsilon^{-pt} f(t)\,dt \qquad (1)$$

where the time function to be transformed f(t) has the period T and p is a complex number. The function f(t) is assumed to be bounded and to have at most a finite number of discontinuities. Appendix A contains a brief explanation of each symbol used in this thesis.

The inverse transform is

$$S^{-1}\left[F(p)\right] = \frac{1}{2\pi j}\int_W \frac{\epsilon^{pt}\,F(p)}{1-\epsilon^{-pT}}\,dp \qquad (2)$$

where F(p) is the function to be transformed and W is the path of the complex integral as shown in Fig. 1. The path W has two separate parts, W_1 and W_2, traversed in the directions indicated by the arrows. The path W must include

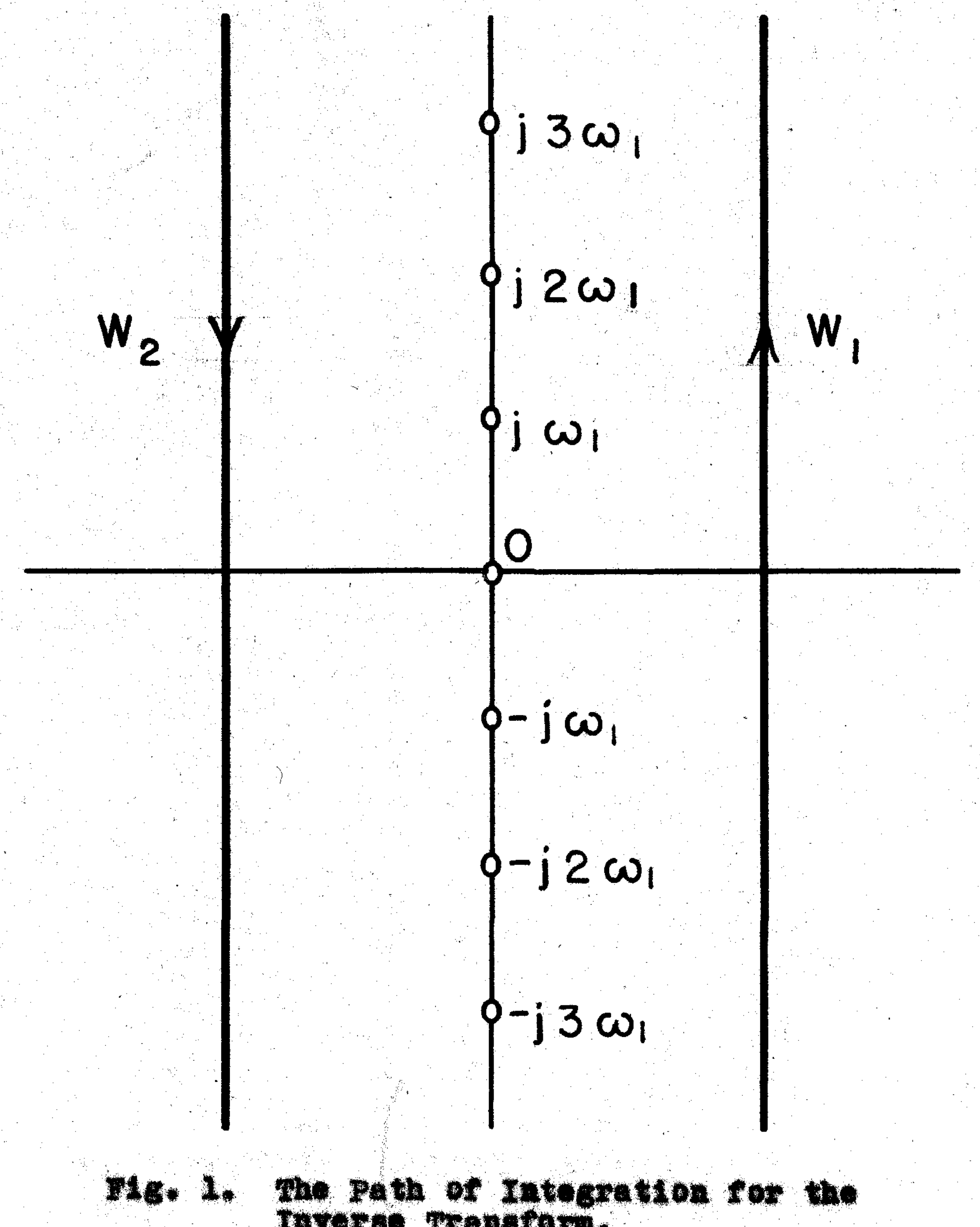

Fig. 1. The Path of Integration for the Inverse Transform.

within it the points $jn\omega_1 = jn\,2\pi/T$ where n is a positive or negative integer or zero. The path W_3 to be used later is the same as W_2 except that the direction is reversed. The two symbols S and S^{-1} indicate the direct transform and the inverse transform respectively. The function F(p) is assumed to be a function analytic everywhere in the finite part of the complex plane except for a finite number of poles as singularities in each finite part of the complex plane. The function F(p) must not have any essential singularities or branch points in the finite part of the plane. This assumption is general enough so that it will include all cases encountered in the particular problem at hand. All of the poles of F(p) must lie to the left of W_2.

The direct transform may be evaluated for a given time function by treating p as a constant and using an ordinary table of integrals. The evalua tion of the inverse transform requires some knowledge of the theory of residues. Either a Fourier series or a sum function may be obtained, but only the sum function was desired for the problem at hand so only this one method of evaluation will be discussed. The inverse transform from (2) may be expressed as two separate integrals:

$$S^{-1}\left[F(p)\right] = \frac{1}{2\pi j}\int_W \frac{\epsilon^{pt} F(p)}{1-\epsilon^{-pT}}\,dp$$

$$= \frac{1}{2\pi j}\int_{W_1} \frac{\epsilon^{pt}F(p)}{1-\epsilon^{-pT}}dp - \frac{1}{2\pi j}\int_{W_3} \frac{\epsilon^{pt}F(p)}{1-\epsilon^{-pT}}dp. \qquad (3)$$

The path of integration W_1 of the first integral of (3) is the same as that of the inverse transform of the ordinary operational calculus and may be evaluated in the same manner. Thus

$$\frac{1}{2\pi j}\int_{W_1} \frac{\epsilon^{pt}F(p)}{1-\epsilon^{-pT}}dp$$

$$= \frac{1}{2\pi j}\int_{W_1} \epsilon^{pt}F(p)dp + \frac{1}{2\pi j}\int_{W_1} \epsilon^{p(t-T)}F(p)dp + \cdots\cdots, \qquad (4)$$

and when $0 < t < T$,

$$\frac{1}{2\pi j}\int_{W_1} \frac{\epsilon^{pt}F(p)}{1-\epsilon^{-pT}}dp = \frac{1}{2\pi j}\int_{W_1} \epsilon^{pt}F(p)dp\,, \qquad (5)$$

which may be evaluated by calculating the residues at all poles to the left of W_1. The second integral of (3) may be evaluated by calculating the residues at the poles to the left of W_3.

The inverse transform is unique, i.e., it has one and only one value for a given value of t, whereas the direct transform is not unique. There is one value of the direct transform that is unique, however, and that is the transform F(p) which has no poles anywhere in the finite part of the complex plane. It is also the transform obtained always from (1) and will henceforth be called the primitive direct transform. Since the direct transform is not uniquely determined, it follows that certain functions of p when substituted in the inverse transform are zero.

B. Circuit Theorems

Three simple theorems are necessary before this method may be applied to the solution of steady-state circuit problems. These are:

1. If $f_n(t) = S^{-1}[F_n(p)]$, $f(t) = S^{-1}[F(p)]$,

   and $F(p) = \sum_{n=1}^{n} F_n(p)$,

   then $f(t) = \sum_{n=1}^{n} f_n(t)$.

This theorem may be inverted if F(p) and $F_n(p)$ are primitive transforms.

2. If $g(t) = g(0) + \int_0^t f(t)\,dt$ and g(t)

is continuous, then

$$S[g(t)] = \frac{1}{p}(1-\epsilon^{-pT})g(0) + \frac{1}{p}S[f(t)].$$

3. If $g(t) = \frac{d[f(t)]}{dt}$ and f(t) is continuous,

then $S[g(t)] = -(1-\epsilon^{-pT})f(0) + pS[f(t)].$

Both the second and third theorems may be applied as many times as is necessary. The initial conditions g(0) and f(0) in the above theorems correspond to the arbitrary constants in the solution using differential equations. When these initial conditions are used in a circuit solution by means of the steady-state operational calculus, they will always be multiplied by a function of p which when substituted in the inverse transform is zero. Hence, the initial conditions always disappear in the final solution, and, in effect, this indicates that the second and third theorems may be written as:

2. If $g(t) = g(0) + \int_0^t f(t)dt,$

then $S[g(t)] = \frac{1}{p}S[f(t)].$ (6)

3. If $g(t) = \frac{d[f(t)]}{dt},$

then $S[g(t)] = pS[f(t)].$ (7)

Essentially this means that in the solution of circuits, the derivative may be replaced by p and the integral by (1/p).

Two other theorems have been found useful in rectifier circuit studies. These are:

1. If $i(t)$ is continuous, then

$$\int_{t_1}^{t_2} \epsilon^{-pt} \frac{di}{dt}\, dt = i(t_2)\epsilon^{-pt_2} - i(t_1)\epsilon^{-pt_1} + p\int_{t_1}^{t_2} \epsilon^{-pt}\, i\, dt. \quad (8)$$

2. If $q(t)=q(t_1)+\int_{t_1}^{t} i\,dt$ and $g(t)$ is continuous,

then

$$\int_{t_1}^{t_2} \epsilon^{-pt} q(t)\,dt = \frac{1}{p}\left[q(t_1)\epsilon^{-pt_1} - q(t_2)\epsilon^{-pt_2}\right] + \frac{1}{p}\int_{t_1}^{t_2} \epsilon^{-pt}\, i\, dt. \quad (9)$$

III. THE SOLUTION OF THE EQUIVALENT CIRCUITS

A. Assumptions

The circuit of the full-wave rectifier with its filter is shown in Fig. 2. During the first half cycle of the applied alternating voltage, tube T_1 conducts and charges condenser C_1 which in turn discharges into the circuit composed of the inductor L, the condenser C_2 and the load resistor R. During the second half cycle tube T_2 conducts to charge condenser C_1 again. The inductor L tends to prevent any change in the discharge current flowing from C_1, and the condenser C_2 tends to prevent the voltage across the load resistor R from changing. Thus the filter composed of L, C_1, and C_2 acts to change the pulsating direct voltage across C_1 into a nearly constant direct voltage across the load resistor. The filter may also be viewed as one of the low-pass type which passes the direct current with very little attenuation but greatly attenuates the higher ripple frequencies present in the pulsating direct voltage across the condenser C_1.

To make a mathematical analysis feasible, certain simplifying assumptions are necessary, and these are listed below.

1. While conducting, the tubes are to have no voltage

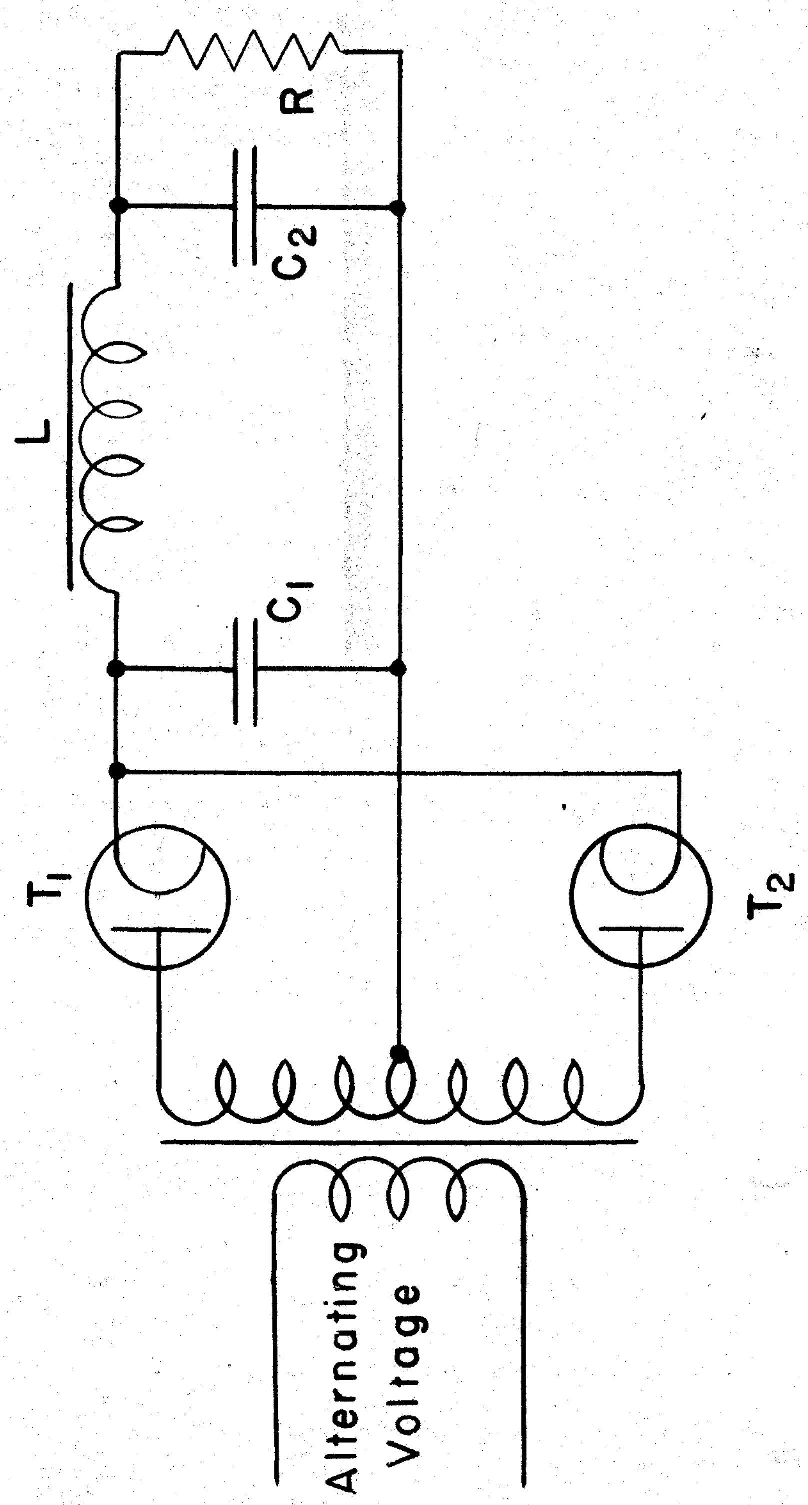

Fig. 2. The Schematic Circuit of the Rectifier and Filter.

drop across them, and while not conducting, their resistance is infinite.

2. When initially not conducting, the tubes start conducting when the voltage across them becomes zero. When initially conducting, the tubes stop conducting when the current through them becomes zero.

3. The voltage source and transformer have no resistance and reactance.

4. The inductor has no resistance.

5. The inductance is constant for all currents and frequencies.

6. The condensers are equal and constant in capacitance and have a zero power factor.

7. The load resistor is constant in resistance and has no inductance or capacitance.

8. The voltage source is sinusoidal and of constant frequency.

9. The output voltage of the transformer is sinusoidal, and the center tap of the secondary gives exactly the same voltage on both halves of the secondary.

10. The steady-state condition is assumed to have been attained.

11. Each tube conducts once during each cycle and conducts for an angle less than or equal to a half cycle.

12. There is no overlap of the conducting periods of the tubes.

The most serious of these assumptions are those listed under 1, 3, and 4, and the errors introduced by these assumptions will in most cases be negligible as long as two conditions are satisfied. These are that the load resistor is substantially greater than the sum of the resistance of one half the secondary winding of the transformer, the resistance of the tube when conducting, and the resistance of the inductor; and that the tube and and transformer resistance and reactance are less than the reactance of the first condenser. When these conditions are not fulfilled, discrepancies should appear between experimental results and the results of this analysis. The errors caused by these assumptions can be reduced to some extent by corrections which will be introduced later. In most practical cases these assumptions will hold fairly well.

B. The General Case

By the use of the assumptions listed above, two equivalent circuits may be set up as shown in Fig. 3. Fig. 3(a) gives the equivalent circuit for the period during which the tube is conducting, while Fig. 3(b) shows the

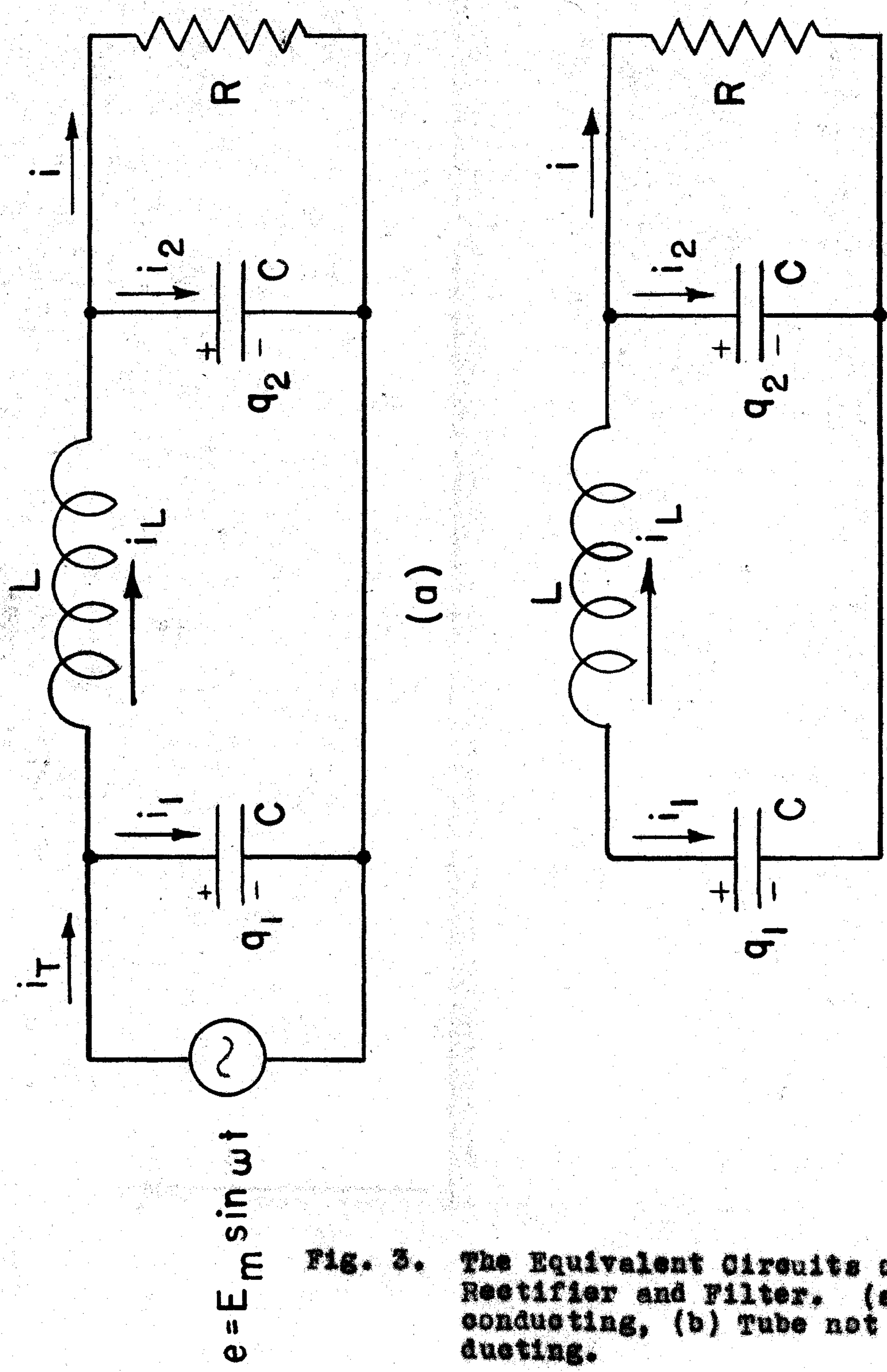

Fig. 3. The Equivalent Circuits of the Rectifier and Filter. (a) Tube conducting, (b) Tube not conducting.

equivalent circuit to apply during the period of no conduction. The equations for the currents and voltages in the circuits were set up and then solved by the use of the steady-state operational calculus. This is outlined in Appendix B. A method for the computation of the conduction angle γ of one tube is presented, and the actual results of the computation for the point a = 0.6 and b = 5.0 are also presented. The solution given in Appendix B shows that the tube angle depends only on the two independent variables, $a = \omega^2 LC$ and $b = \omega CR$, where $a = \omega^2 LC$ is the ratio of the reactance of the filter inductor to the reactance of one filter condenser and $b = \omega CR$ is the ratio of the load resistance to the reactance of one filter condenser. It is shown furthermore that under the assumptions made all of the characteristics of this rectifier circuit depend on these two variables alone.

It was found that for values of a and of b somewhat greater than five greater accuracy was required than that afforded by the use of the slide rule. This would have meant that a prohibitively large amount of time would have been consumed in calculating each point. The values of a and of b used were for this reason confined to five and below except for the boundary points where either or both a and b are assumed infinitely large. Filter resonance seems to occur at a = 0.5 for at that value of a the sum of the reactances of the two filter condensers and of the inductor is zero. These reactances should be calculated using the fundamental ripple frequency which is twice the supply frequency. While the calculation of several points was attempted for a = 0.5, no complete solution was obtained because no value of γ could be found that would make $\beta_1 = \beta_2$ in equations (B-28) and (B-30), where (B-28) and (B-30) indicate equations (28) and (30) in Appendix B. It may be that the correct value of γ has been overlooked, or what is more likely, that the solution no longer holds for resonance. The calculation of one point, that of a = 0.25, b = 1.0, has, however, been completed, showing that a solution is obtainable for smaller values of a. Since no solution has been obtained for a = 0.5 and since the problems of practical importance at present have a always greater than 0.5, the lowest value of a used in the analysis was 0.6.

C. Special Cases

There are two special cases of this general solution which, it was decided, would be better to consider separately since the solutions of the special cases differ in many details from the general solution. One of these special cases is that for $a \to \infty$. This might be considered the case of an infinitely large inductor in which the current flowing through it is constant. One method of attacking this particular special case would be to allow $a \to \infty$ in the general solution and to evaluate the resulting equations. This was not done because the solution for this special case is quite a bit simpler than the general solution and it was obtained before the general solution was finished. The outline of the solution for this special case is presented in Appendix C, and it will be noticed that differential equations were used in the solution rather than the steady-state operation calculus because the operational calculus had not been developed at that time. Later a was allowed to approach infinity in the general solution, and when the resulting indeterminant expressions were evaluated, the solution as obtained from the general solution was found to agree with the solution obtained earlier. It is shown in Appendix C that the tube conduction angle γ is a function of the one parameter $b = \omega CR$ for this special case.

The other special case to be considered here is the case where each tube is conducting 180 degrees which will be called the non-cut-off case. This name was adopted because current in the tube is not cut-off or stopped by the action of the particular type of filter circuit used. The current in each tube is stopped rather by conduction starting in the other tube, and a form of commutation between the tubes is set up in which the current switches from one tube to the other and back again. The main problem for this special case is not that of finding γ which is known to be 180 degrees, but that of finding the boundary or dividing line between the cut-off and non-cut-off cases. This can be accomplished by solving for the values of a and b which will make the tube current i_T zero at one point and positive everywhere else. The leading current taken by the first condenser in the filter requires that the point of zero tube current occur at t = T. This yields one equation involving a and b, and if a value of a is specified, the corresponding value of b may be calculated. The outline of this method is presented in Appendix D.

IV. THE CHARACTERISTICS OF THE RECTIFIER CIRCUIT

A. The Tube Angles, γ and β

The characteristics of the circuit that are of the most use in the design of the circuit and in predetermining its properties are the tube angles, the output average voltage, the ripple, the peak and average tube currents, and the peak inverse voltage on the tubes. The calculation of these various characteristics have been presented in the several appendices, i.e., Appendix B for the general case, Appendix C for the special case of $a \rightarrow \infty$, and Appendix D for the non-cut-off case.

The angle γ during which the tube conducts is presented in Fig. 4 as a function of $a = \omega^2 LC$ for various values of $b = \omega CR$. For a fixed value of $\omega^2 LC$ this angle varies from zero degrees at open-circuit ($\omega CR \rightarrow \infty$) to 180 degrees (non-cut-off case) for values of ωCR in the neighborhood of 0.6. For smaller values of ωCR down to zero (short-circuit), γ is equal to 180 degrees. With ωCR fixed, the angle γ varies little for values of $\omega^2 LC$ above 10, but for $\omega^2 LC$ less than 10, the variation becomes considerable especially as filter resonance is approached ($\omega^2 LC = 0.5$). The angle β at which the tube stops conducting is presented in Fig. 5

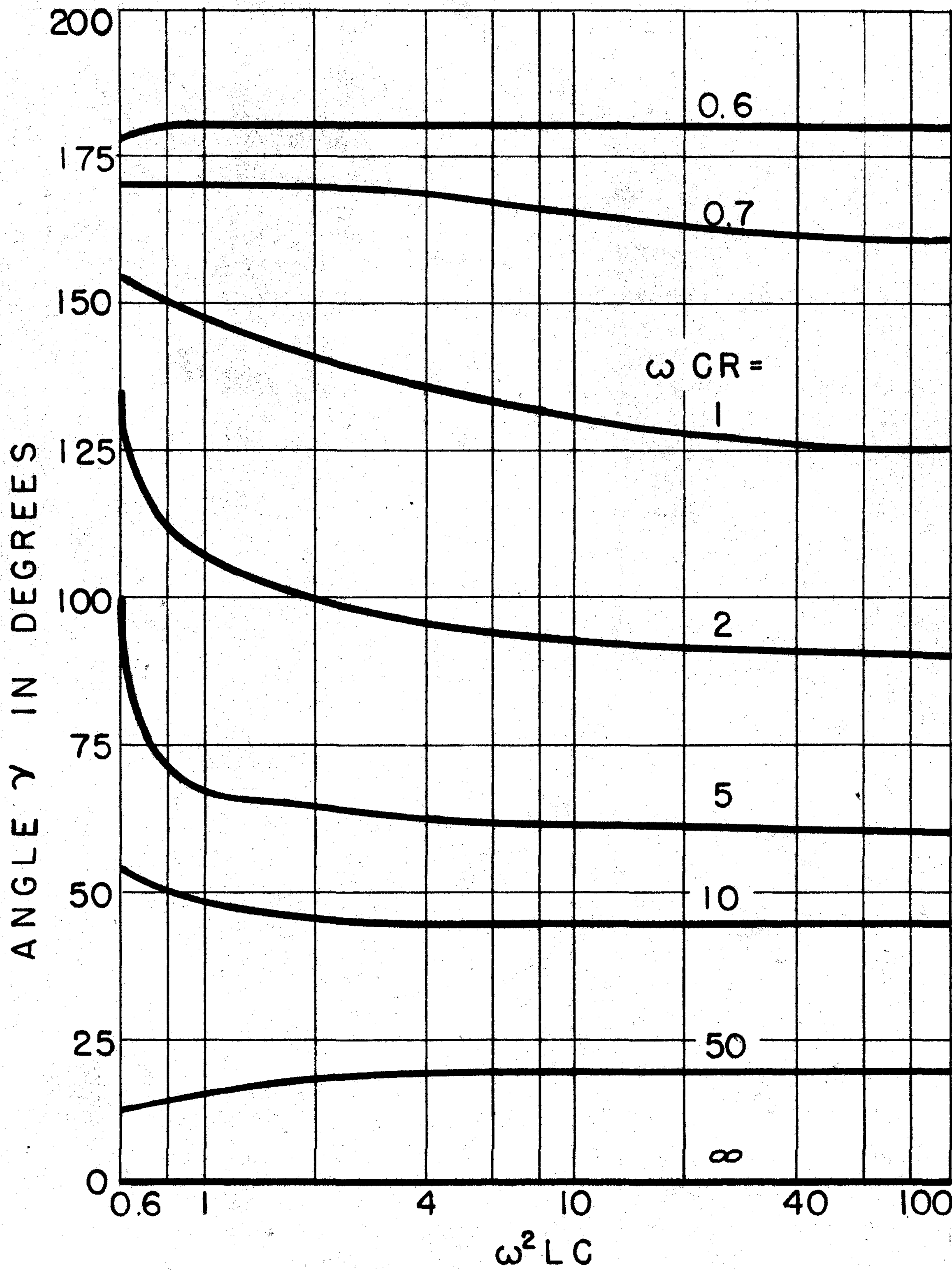

Fig. 4. The Angle γ during which the Tube Conducts.

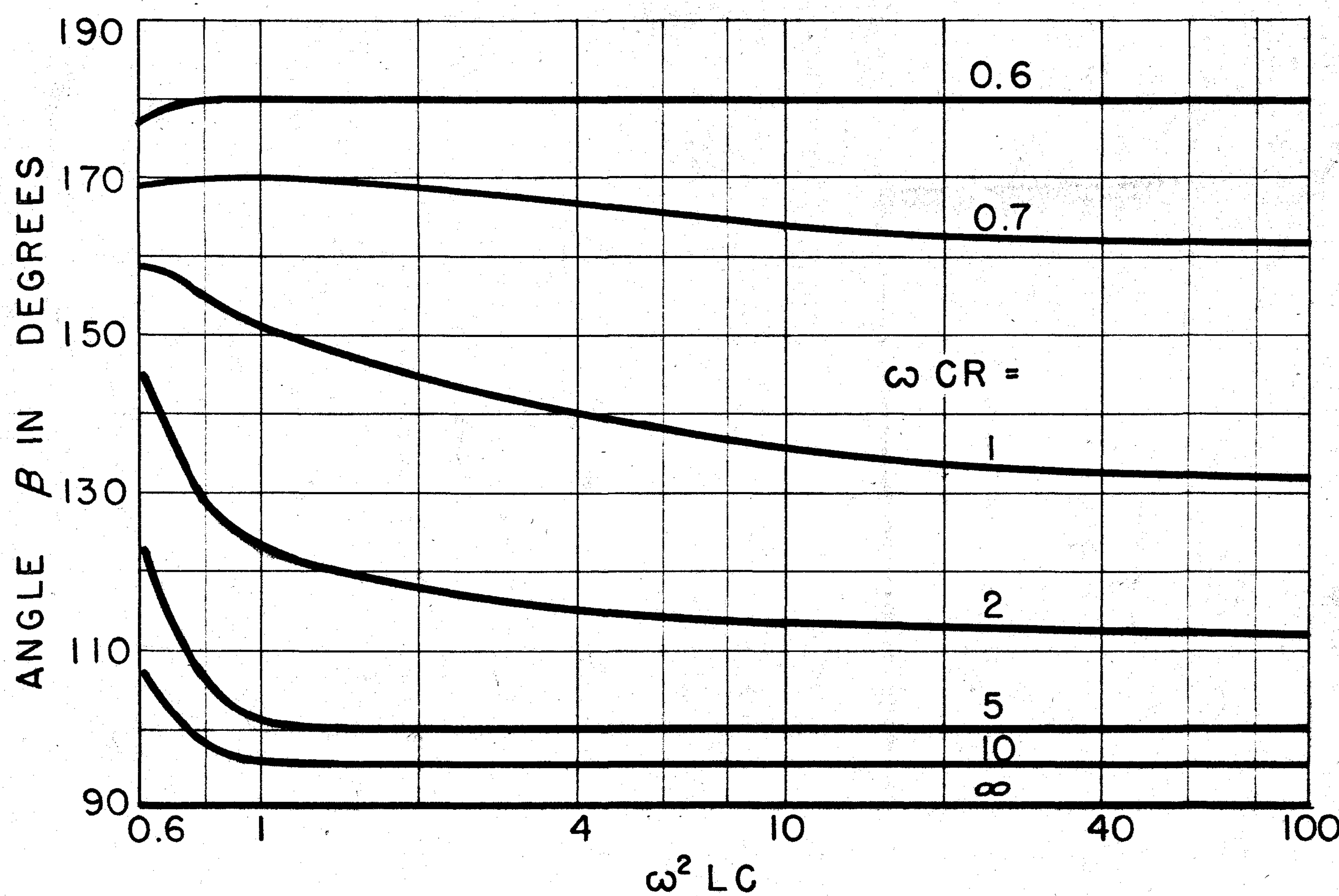

Fig. 5. The Angle β at which the Tube Stops Conducting.

in a similar manner to that used for γ. For a fixed value of $\omega^2 LC$, β varies from 90 degrees at open-circuit to 180 degrees for values of ωCR in the neighborhood of 0.6. For smaller values of ωCR, the non-cut-off condition occurs, and β is equal to 180 degrees. The variation with ωCR fixed is similar to that for γ. The angle α at which the tube starts conducting may be found by the use of the relation, $\alpha = \beta - \gamma$. Increasing the load of the rectifier circuit has the effect of increasing both angles β and γ, while α is decreased. The effect of approaching resonance in the filter is similar in that angles β and γ are increased, while α is decreased.

B. Average Output Voltage

One of the most important characteristics of a rectifier circuit is the average output voltage. This voltage may be obtained from the ratio of the average output voltage to the maximum value of the sinusoidal voltage across one-half of the transformer secondary, E_{dc}/E_m. The ratio E_{dc}/E_m is given in Fig. 6 as a function of $\omega^2 LC$ for various values of ωCR. The zero was suppressed so that the curves would be more useful in design work. For a fixed value of $\omega^2 LC$, the ratio decreased from a maximum of unity for $\omega CR \rightarrow \infty$ (open-circuit) to 0.6366 for ωCR equal approximately to 0.6.

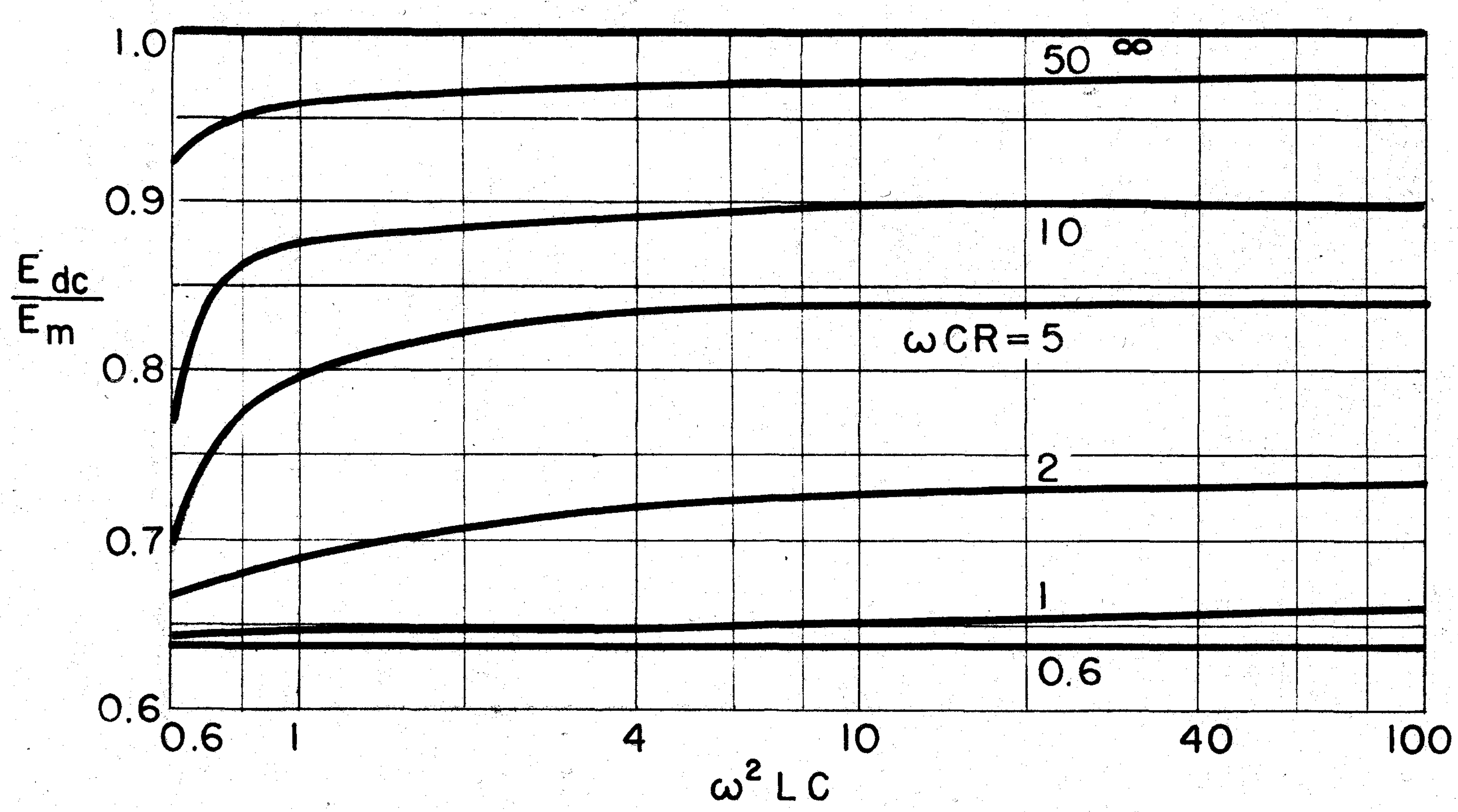

Fig. 6. The Average Output Voltage.

For values of ωCR from 0.6 to zero (short -circuit) the non-cut-off case applies, and $E_{dc}/E_m = 0.6366$. For a fixed value of ωCR, the ratio E_{dc}/E_m varies very little for $\omega^2 LC$ larger than 10, but for smaller values of $\omega^2 LC$, the ratio changes considerably especially as $\omega^2 LC$ approaches the filter resonance value of 0.5. In general the greater the angle γ through which the tube fires, the closer the ratio E_{dc}/E_m approaches the non-cut-off value of 0.6366. Similarly the smaller the angle becomes, the closer E_{dc}/E_m approaches the no-load value of unity. Again both increasing the load on the rectifier circuit and approaching filter resonance have a similar effect, that of making the ratio E_{dc}/E_m approach 0.6366 more closely.

C. Ripple Voltage

The ripple voltage across the load resistor of the rectifier circuit is another important characteristic. It may be obtained from the per cent ripple r which is defined as 100 times the ratio of the effective value of the fundamental ripple voltage appearing across the load resistor to the average voltage E_{dc} across the same resistor. The reason for using only the fundamental ripple voltage is that this type of filter attenuates the higher ripple frequencies so much more than the fundamental that in practice only the fundamental is sufficiently large to warrant consideration. Since this circuit is a full-wave type, the fundamental ripple frequency is always twice the supply frequency. As given in Appendix B the per cent ripple was calculated across the first filter condenser by the use of the Fourier series of the condenser voltage, and the per cent ripple across the load resistor was found by multiplying the per cent ripple across the first condenser by the gain in the filter at the fundamental ripple frequency. It was found that for large values of a and of b, the per cent ripple across the first condenser when plotted against the variables (1/a) or (1/b) produces nearly straight lines. The empirical equations of these straight lines were found and when multiplied by the attenuation of the filter, gave an expression for the per cent ripple across the load resistor. This expression is

$$r = \frac{31.5 + 67.25a}{a\sqrt{4a^2 + b^2(4a-1)^2}}, \qquad (10)$$

and the error in the per cent ripple will be less than five per cent if $a \geq 2$ and $b \geq 5$. The empirically derived constants are 31.5 and 67.25. The equation (10) will be applicable in many of the problems encountered in practice particularly those in which the circuit is lightly loaded. It expresses the per cent ripple for the region where each tube is firing only a short time, the output voltage is high, and the charges on the filter condensers do not vary much. Hence, the per cent ripple in this case will only be a few per cent at most. As the rectifier circuit is loaded more heavily, the tube angle increases, and the amount of charge supplied to and taken from the filter condensers twice each cycle becomes an appreciable amount of the total charge on the condensers. This means a high per cent ripple, and instead of presenting an empirical equation for this region, the per cent ripple is shown as curves in Fig. 7. For clarity in presenting the curves, ωCR has been used as the abscissa, while the curves are drawn for constant values of $\omega^2 LC$. For $\omega^2 LC$ constant, the ripple increases from zero for $\omega CR \rightarrow \infty$ (open-circuit) to a maximum in the vicinity of ωCR equal to unity and then approaches zero again as ωCR approaches zero (short-circuit). For a fixed value of ωCR, the per cent ripple increases from zero for $\omega^2 LC \rightarrow \infty$ (infinitely large inductance) toward a maximum as $\omega^2 LC$ approaches 0.5 (filter resonance). When ωCR is less than approximately 0.6, the non-cut-off case applies, and the per cent ripple equation is that of (D-5). For the non-cut-off case, the per cent ripple

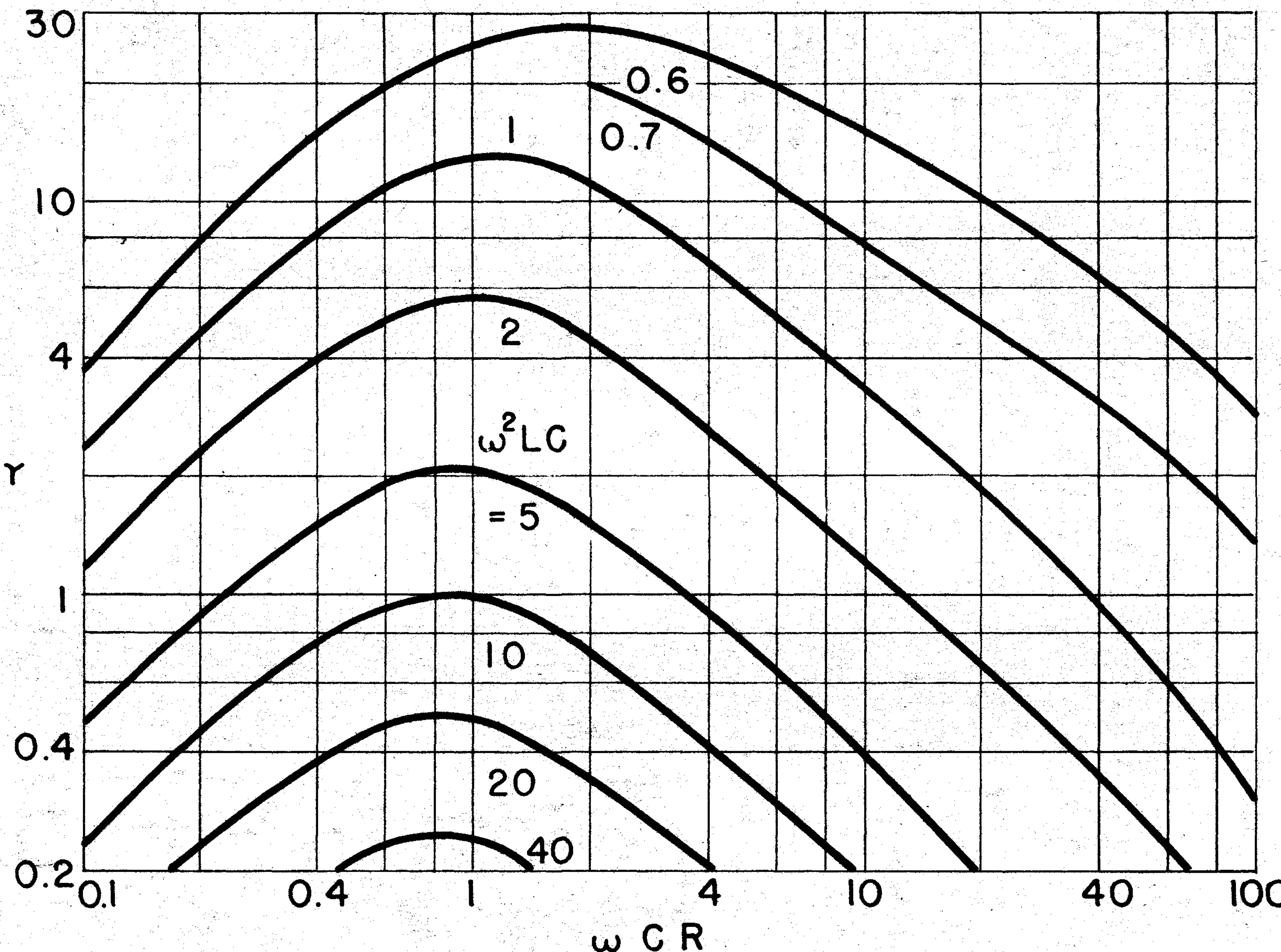

Fig. 7. The Per Cent Ripple Voltage.

across the first filter condenser is 47.14 per cent, and this ripple is multiplied by the gain in the filter which approaches zero as ωCR approaches zero (short-circuit). The reason for this is that for low values of ωCR most of the ripple voltage appears across the inductor and very little of it appears across the output load resistor, while all of the average voltage appears across the load resistor.

D. Peak Tube Current and Voltage

In choosing the rectifier tubes or in determining how much current a given rectifier tube can deliver and still stay within its rated values, the peak tube current required by the action of the filter circuit and load resistor is needed. This peak tube current may be obtained from the ratio P_T of the peak tube current to the average current for one tube. The values of this ratio for various values of ωCR and $\omega^2 LC$ are given in Tables 6,7 and 8. It may be noticed from these tables that for a given value of ωCR, the value of P_T increases slightly as $\omega^2 LC$ increases from 0.6, approaches a maximum, and then decreases slightly as $a \rightarrow \infty$. The total variation is small and for practical purposes may be neglected. It is sufficient then to use the values of P_T for one value of a, and the one chosen was the special case of $a \to \infty$ as given in Table 7. When these values of P_T were plotted against $b = \omega CR$ on log-log graph paper, the points fell on a straight line as long as $b \geq 0.6$. The equation for this line was found to be

$$\log_{10} P_T = 0.5025 \log_{10} b + 0.7004, \qquad (11)$$

and this may be used to determine P_T as long as $b \geq 0.6$. For values of $b < 0.6$, i.e., the non-cut-off case, equation (C-16) may be used. The ratio P_T increases from a value of two for b equal to zero and approaches infinity as b approaches infinity. For b equal to zero, the tube current is rectangular in shape, and hence the peak current is just twice the average current. As b increases, the tube current becomes more and more peaked with the peak occurring at the time of starting of conduction in the tube. For a peaked current of this type the average value is very small, and the ratio of peak to average current very large.

The inverse peak voltage applied to each tube should be known in choosing the tubes. For this circuit this peak voltage is always $2E_m$. This is twice the peak alternating voltage appearing across one half the transformer secondary.

Sometimes it becomes necessary to obtain values of the characteristics outside of the ranges presented in the curves and equations. These values may be obtained by plotting values taken from the curves and equations along with values for the end points such as those for $a \rightarrow \infty$, against one of the independent variables a, b, (1/a), or (1/b). This method is useful also to interpolate for values located between the curves presented here. The characteristic curves have been plotted in such a fashion that they do not depend in any way on the value of a particular parameter, such as a particular frequency, and thus are equally as applicable at other frequencies than the sixty-cycles-per-second frequency used in the experimental tests. This, of course, presupposes that the assumptions are still tenable.

V. EXPERIMENTAL WORK

A. Preliminary Experiments

Early in the research work for this thesis it was realized that the problem could be solved wholly mathematically, wholly experimentally, or as a combination of the two methods. It was realized, too, that the most satisfactory method would be a combination of both experimental and mathematical analysis, and this was used.

Preliminary experiments were made with the circuit to observe the angle of conduction of the tubes. The measurement of the inductor in the filter was done but with no attempt for any great accuracy because the preliminary experiments were for more of a qualitative observance of the operation of the circuit than a quantitative one. For example, no attempt was made to observe the effect of any direct current in the inductor or the effect of varying the amount of alternating current. The rated capacitance of the condensers was used without any attempt to measure it. These experiments showed, however, that a variation of ωCR produced a greater change in the characteristics of the circuit than a similar change in $\omega^2 LC$. The angle of conduction γ in the tube was observed to vary from zero degrees for an open-circuit load ($\omega CR \rightarrow \infty$) to 180 degrees for ωCR small. The phenomenon of the change from the cut-off to the non-cut-off type of operation was observed, and it was realized that the non-cut-off case would have to be studied separately. The effect of filter resonance was also observed, but it was not realized at the time that the mathematical solution might break down for the region in which filter resonance occurs.

In the mathematical solution an estimate of the value of γ was needed to shorten the labor of calculation, and this estimate could be found from the results of the preliminary experimental work. This work also facilitated the finding of several mistakes in the derivation of the equations for the mathematical solution. For example, one of the first attempts at calculating a point showed that the solution would occur at a value of the angle γ in the neighborhood of 50 degrees, while the experimental work showed that it should occur in the neighborhood of 130 degrees. Upon re-examination and checking the derivation, a mistake in sign in one of the equations was found. When the equations were corrected, a value of γ in the neighborhood of 130 degrees was found to solve the equations.

A more exact experimental analysis of the circuit was made later for two reasons, first, to verify the calculated results and to determine at the same time where the experimental results would not agree very well with the calculated results, and second, to help in determining what points should be selected for the calculated values. The principal items that should be measured before setting up the circuit are the inductance of the filter inductor, the capacitance of the filter condensers, and the tube resistance.

B. Measurement of the Filter Inductor

The inductor chosen was a Western Electric Type 107A Retard Coil and was chosen because of its high inductance and low resistance. Since it has an iron core, the inductance will vary with the amounts of alternating current and of direct current flowing in the inductor. In some experiments the previous magnetic history of the core must be erased by a complete demagnetizing process, but this was not found necessary here. Meier and Waidelich[5] described a method of measuring the inductance using a modification of the Owen bridge, and this method was applied in this case. This allowed introduction of a direct current as well as an alternating current in the inductor being measured. The alternating current should have the same frequency as the fundamental ripple voltage. Since the fundamental ripple voltage is twice the supply frequency and the supply frequency to be used was sixty cycles per second, an alternating voltage source of 120 cycles per second was necessary. This was obtained by using a wound-rotor induction motor as an induction generator with the rotor driven at synchronous speed in the opposite direction to the rotating magnetic field of the stator. The 120-cycle-per-second voltage then appeared at the terminals of the rotor if the stator was excited with 60-cycle-per-second alternating current. The driving motor for the rotor was a direct-current shunt motor although this was not essential.

The fundamental bridge diagram is shown in Fig. 8 where L and R_L represent the apparent inductance and apparent resistance of the inductor. In an inductor with an iron core the resistance to alternating current is quite different from the resistance to direct current because of the effect of hysteresis losses principally. It is this resistance to alternating current that will be indicated as R_L and will be called the apparent resistance. The resistors R_1, R_2, and R_4 were three decade resistance boxes with R_1 and R_2 chosen so that they would carry the full value of direct current and alternating current used in measuring the inductor. The condensers C_3 and C_4 were two decade capacitance boxes. The other two condensers C_5 and C_6 were by-pass and blocking condensers and had suffi-

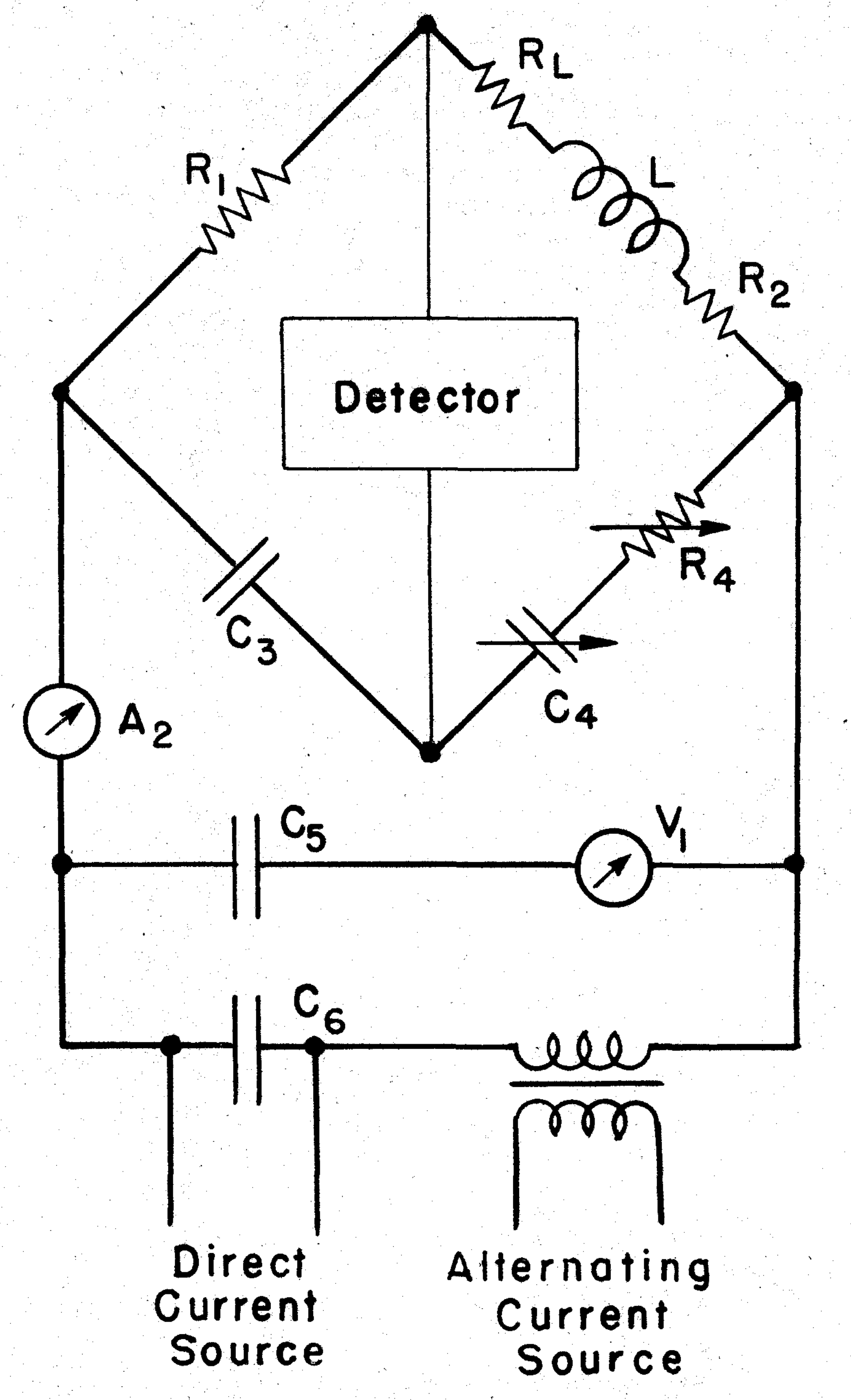

Fig. 8. The Bridge Circuit Used to Measure the Filter Inductance.

cient capacitance so that they would introduce negligible reactance. The meter V_1 was used to measure the alternating voltage applied to the bridge, while the meter A_2 was a direct-current ammeter used to measure the direct current flowing in the inductor. The transformer coupling the alternating-voltage supply to the bridge had a low enough resistance so that not much direct voltage appeared across it and was operated well below its rated capacity so that no distortion of the applied sine-wave alternating voltage was caused by the direct current flowing in the transformer. The detector used was a cathode-ray oscilloscope, and the alternating-voltage source was monitored by another cathode-ray oscilloscope to see that its frequency and wave form did not shift. In some cases the resistance R_2 is necessary to obtain enough resistance for a sensitive bridge balance, but in this case it was found that R_2 could be left zero.

The balance equations for the bridge gave the following expressions for the inductance and for the apparent resistance of the inductor:

$$L = R_1 R_4 C_3 \tag{12}$$

$$R_L = \frac{R_1 C_3}{C_4} - R_2. \tag{13}$$

The direct current I_{DC} flowing through the inductor was measured by the meter A_2, while the alternating current I_{AC} was calculated from the alternating-voltage reading V_1 by using

$$I_{AC} = \frac{V_1}{\sqrt{(R_1+R_2+R_L)^2+(\omega L)^2}} . \qquad (14)$$

The bridge was set up along with the power supplies and detector, and the following values were chosen: R_1 = 100 ohms, R_2 = 0 ohms, C = 0.705 mfd., and frequency = 120 cycles per second. For greater sensitivity R_1 should have been higher, but this was the highest value that could be used and still stay within the current rating of the decade box. Actually, the sensitivity was quite good enough for the purpose at hand. The applied alternating voltage was varied, and the direct current was zero for the first part of the test. The results of this part of the test are shown in Table 1. For each value of the alternating voltage V_1, the bridge was balanced and readings taken of the resistance R_4 and the capacitance C_4. The inductance L, the apparent resistance R_L, and the alternating current I_{AC} were then calculated by means of (12), (13), and (14) respectively. The second part of the test consisted in increasing the direct current from zero to 200 milliamperes which was approximately the maximum direct current that was to be used in the filter circuit, and then repeating the first part of the test. The results of the second part are given in Table 1 also.

TABLE 1

Inductance Measurement of the Western Electric

Type 107A Retard Coil

V_1 volts	I_{DC} milli-amperes	C_4 micro-farad	R_4 ohms	L henries	R_L ohms	I_{AC} milli-amperes
29.15	0.0	0.390	102,400	7.225	180.7	5.34
58.0	0.0	0.380	104,500	7.360	185.4	10.45
86.4	0.0	0.380	105,800	7.460	185.4	15.35
115.4	0.0	0.380	106,600	7.510	185.4	20.41
146.8	0.0	0.380	107,400	7.580	185.4	25.65
179.5	0.0	0.380	108,000	7.610	185.4	31.25
209.6	0.0	0.377	108,500	7.650	187.0	36.30
239.5	0.0	0.376	108,800	7.680	187.4	39.95
29.6	200.0	0.440	101,500	7.160	160.2	5.55
58.6	200.0	0.400	103,400	7.290	176.2	10.79
91.4	200.0	0.390	104,900	7.400	180.7	16.60
120.9	200.0	0.390	105,700	7.450	180.7	21.82
148.6	200.0	0.390	106,400	7.500	180.7	26.63
180.4	200.0	0.390	106,900	7.540	180.7	32.17
210.2	200.0	0.390	107,400	7.580	180.7	37.25
238.3	200.0	0.390	107,800	7.610	180.7	42.20

The variation in the inductance L was small, and the inductance increased with increasing alternating current and decreased with increasing direct current. The range in alternating current used was somewhat more than that expected to be encountered in the actual filter circuit operation. The apparent resistance R_L was small enough compared to the reactance of the inductor so it would not affect the action of the filter to any appreciable extent. The direct current resistance of the inductor was found to be 33.18 ohms, and it is interesting to notice that the apparent resistance was approximately five times as great as this direct current resistance.

In using the rectifier filter circuit as an experimental check on the calculated work, it was decided to vary ωCR with $\omega^2 LC$ fixed at one value. This necessitated choosing a value of L from the data presented in Table 1 and assuming that L did not change from this value. For 220 volts alternating voltage applied to the rectifier circuit from one-half of the transformer secondary, the maximum fundamental ripple voltage appearing across the first filter condenser is 93.4 volts. If it is assumed that all of this ripple voltage appears across the inductor, the ripple current would be approximately 16.8 milliamperes. It was decided to use the value of the inductance for an alternating current of one-half of 16.8

or 8.4 milliamperes and for a direct current of 100 milliamperes as representative average values. The selected value of inductance was then 7.27 henries assuming that the inductance decreased linearly as the direct current increased which was found to be a good first approximation to the actual variation.

C. Measurement of the Filter Condensers

The value of ω^2LC to be used in the experimental test was selected as 2.0 to facilitate comparison between experimental and calculated results and to allow enough range of ωCR so that the non-cut-off case could be reached. This meant for a supply frequency of sixty cycles per second, that two capacitances of 1.935 microfarads had to be made up by combining available condensers. A general purpose bridge[2] was set up with each of the two ratio resistance arms containing a decade resistance box, one of the other two arms contained a decade capacitance box and a decade resistance box, and the other arm contained the condenser to be measured. The source of voltage was a beat-frequency oscillator, and the detector was a cathode-ray oscilloscope. When the bridge was balanced, not only the capacitance of the unknown condenser but the series resistance as well could be calculated. Various combinations of the standard condensers available were made and measured, and the two combinations finally selected had capacitances within one per cent of the value desired. The first condenser combination had a capacitance of 1.925 microfarads and a series resistance of 7 ohms, while the second combination had a capacitance of 1.943 microfarads and a resistance of 12 ohms. Both of these resistances were small compared to the reactances of the condensers at the fundamental ripple frequency.

D. Measurement of the Tube Resistance

The assumption of zero tube resistance when conducting was necessary to make the solution of the rectifier and filter circuit feasible, but it imposed a rather severe restriction on the agreement of experimental and calculated results for small values of ωCR. It was found by means of experiment, however, that a simple and approximate method might be used to take care of the effect of both the tube resistance and the direct-current inductor resistance. It was decided in the beginning of the experimental work that since the effect of the tube resistance and inductor resistance was unknown, both should be made small without going to extremes. Thus it was decided to put two diodes in parallel if they were in the same envelope but not to put several tubes in parallel to obtain an extremely low resistance. Mercury-vapor tubes will give a very low voltage drop during conduction, but they were not used for two reasons, first, they are almost never used in practice with this type of filter circuit, and second, they do not exhibit the almost constant resistance characteristic of vacuum tubes. The mercury-vapor tubes have an almost constant voltage drop during conduction and thus have a resistance that varies widely with the current flowing in the tube.

The diode vacuum tubes when conducting have a current flowing through them which varies very nearly as the voltage across the tube raised to a power. The exponent involved is usually between one and two with most values in the vicinity of 1.5. The resistance of the tube thus varies but only slightly as compared to the great variation of resistance encountered in the mercury-vapor tube. There is, however, a question of what definition of resistance should be used when applied to the tube. For example the resistance to a small alternating current superimposed on a sizeable direct current will be practically the inverse slope of the tube current-voltage characteristics at the point of the direct current, while the resistance to the direct current itself will be the direct voltage across the tube divided by the direct current. These two quantities may differ quite a bit depending on the tube and the point of operation. Since in actual operation in this circuit, the tube current increases from zero to a maximum and then decreases to zero again, it was decided to use the resistance to direct current, i.e., the total direct voltage across the tube divided by the direct current flowing through the tube as the closest approximation to the changing resistance of the tube. The only point that remained to be defined was at what value of direct current should this resistance be specified. It was shown in the characteristics section that the ratio of peak to average values of tube current varied from very large amounts near open-circuit to two at short-circuit. Most of the disagreement between experimental and calculated results, it will be found, occurred near short-circuit, i.e., when the load resistance was the same order of magnitude as the sum of the tube and inductor resistances. With these facts in mind, it was decided to define the tube resistance at the point where the current in the tube was equal to the average current through the tube during one cycle.

There was one additional complication in this particular experimental test, and that was that the average current in the tube will vary from zero for open-circuit to 100 milliamperes for near short-circuit. It was decided to take 100 milliamperes as the point at which the resistance was to be calculated since for lighter loads the peak to average tube current increased appreciably. Upon looking over the commercially available tubes, it was found that a

TABLE 2

The Voltage-Current Characteristic of a Type 83-V Tube

Two Diodes in Parallel

Voltage across the tube	Current through the tube in milliamperes
14.03	175.0
12.01	151.0
10.62	126.0
8.88	97.5
7.31	75.5
5.40	51.4
2.86	25.9
1.57	14.7

group of newer diodes made especially for use with condenser-input filters had resistances that varied from about 175 ohms to 100 ohms. Most of these tubes have only one diode in one envelope. The older group of rectifier tubes all of which had two diodes in one envelope, had resistances that varied from about 500 to 150 ohms per diode. From the latter group the tube type 83-V was chosen as the one to be used, and the two diodes were placed in parallel. Several of the tubes were measured to match their resistances, and a typical set of data showing the variation of current through the tube with voltage across the tube is presented in Table 2. The averaged resistance of the two selected 83-V tubes was 91.7 ohms.

E. Consideration of Other Factors

There are several more minor points which had to be considered before the test was performed. The most important of these was the resistance and leakage reactance of the transformer and of the source supplying the transformer. A transformer and a source both of large capacity were used to keep this resistance and reactance low so that it might be neglected. The resistance of transformers used in practical rectifiers usually cannot be neglected, but it can be taken care of by including it with the tube resistance. This transformer resistance should be that of one half of the secondary. The reactance of the transformer cannot be compensated for this way, and an extension of the analysis would be necessary to include it. The voltage unbalance in the secondary of the transformer was found to be negligible, and the alternating voltage was measured across one-half of the secondary. There may have been an impedance unbalance in the transformer secondary, but since the impedance was considered small enough to be neglected, the unbalance was neglected also. The frequency was measured several times and was found to be within one per cent of sixty cycles per second at all times during the test so that no correction was made for it. The voltage wave form on the secondary of the transformer was observed on the cathode-ray oscilloscope several times during the test and was a sine wave within the limits of judgment by visual observation. An appreciable amount of higher harmonics in the line voltage would have affected the current into the first filter condenser and thus affected the starting and stopping angles in the tubes. For this reason an analysis of the line voltage was made by the General Radio Wave Analyzer, and the result was that no harmonic voltage large enough to produce an appreciable change in the first condenser current was found.

The leakage current in the filter condensers could be neglected because paper condensers were used. In most practical cases, electrolytic condensers with appreciable leakage current are used. To take care of this, the leakage current should be added to the load current when the output load resistance is calculated. Some rectified voltage was contributed by the fact that the tubes used had filaments supplied with alternating current and that during parts of the cycle part of the filament is negative with respect to the plate causing electrons to pass to the plate. With no alternating voltage applied to the transformer, the direct voltage caused by this effect was of the order of a volt or two and thus was negligibly small compared to the voltages to be used. This effect would be much smaller if tubes with indirectly-heated cathodes had been used. The measurement of the load resistors used was accomplished by using a high-resistance direct voltmeter and direct-current milliammeter with the milliammeter connected in such a way as to measure the current taken by the voltmeter. Since the voltage drop across the milliammeter was a small part of a volt, the error caused by this drop was neglected. The tubular wire wound resistors used as the load have a very small reactance at the ripple frequency of 120 cycles per second.

F. The Experimental Test

The circuit was set up as shown in Fig. 2 with the various meters added to this basic circuit. An alternating voltmeter was placed across one-half of the transformer secondary to measure the effective value of the applied voltage. A one-ohm resistor was placed in each half of the transformer secondary winding, right next to the center tap so that the tube angles could be observed by means of a cathode-ray oscilloscope. A direct voltmeter and a wave analyzer were placed in parallel across the load resistor to measure the average voltage and the ripple voltage at the load. A direct current milliammeter measured the current

TABLE 3

Test of the Rectifier Circuit for $\omega^2 LC = 2.0$

Applied Alternating Voltage	I milli-amperes	E_{dc} volts	Effective Ripple Voltage	γ degrees	ωCR	E_{dc}/E_m	r per cent
225.6	3.16	305.5	0.59	17.14°	70.45	0.9575	0.1933
225.7	7.42	299.0	1.34	26.78°	29.40	0.9370	0.4480
224.4	17.25	285.0	2.82	40.73°	12.06	0.8980	0.989
224.5	22.7	276.5	3.71	51.42°	8.885	0.8717	1.342
224.2	41.0	250.3	5.80	66.27°	4.455	0.7897	2.317
225.3	80.5	215.7	9.58	97.30°	1.953	0.6575	4.444
224.2	182.0	182.3	11.87	140.40°	0.731	0.5755	6.506
113.2	40.7	107.7	4.97	99.30°	1.929	0.6728	4.618
113.0	71.9	94.7	5.76	140.90°	0.960	0.5927	6.082
113.0	110.5	87.8	5.52	180.00°	0.5793	0.5495	6.285
113.1	137.5	84.7	4.88	180.00°	0.4492	0.5295	5.765
113.0	187.7	79.7	3.98	180.00°	0.3097	0.4990	4.990

taken by the load resistor, the direct voltmeter, and the wave analyzer in parallel. The results of this test at two input voltages of 220 and 110 volts are shown in Table 3.

The second voltage (110 volts) was used when it became apparent that with 220 volts it was not possible to go into the non-cut-off region far enough and still stay within the maximum current values for the tubes. The first five columns of Table 3 are the experimentally obtained data, and the last three columns are the calculated results from these data. The product ωCR was obtained by multiplying 2π, the applied frequency which was sixty cycles per second, the averaged capacitance of one filter condenser which was 1.935×10^{-6} farads, and the load resistance R in ohms which was equal to E_{dc}/I. The voltage ratio E_{dc}/E_m may be found by dividing E_{dc} by E_m which was $\sqrt{2}$ times the applied alternating voltage. The per cent ripple r was obtained by dividing 100 times the effective ripple voltage by the load voltage E_{dc}.

The tube angle γ was included in the experimental data because of its value in indicating how much the circuit has been loaded and approximately when the non-cut-off border line was reached. This border line was reached at ωCR equal approximately to 0.6 as was predicted in the calculations. The experimental values of the angle γ had their chief use in obtaining an estimate of γ for the calculations. The experimental and calculated values of γ agreed very closely, but no extensive comparison was presented here because it was felt that the comparison of the average voltage ratio and per cent ripple values were far more important. What discrepancies existed between calculated and experimental values of γ occurred at the higher values of tube current. For example the observed value of 140.4° for γ at $\omega CR = 0.731$ and at an average current of 182.0 milliamperes has a much larger discrepancy than the observed value of 140.9° for γ at $\omega CR = 0.960$ and at an average current of 71.9 milliamperes. No observable amount of overlap between the tubes in the non-cut-off region was found, and it is believed that the small amount of leakage reactance in the transformer prevented any sizeable overlap angle from appearing.

G. The Results for the Average Output Voltage

The experimental data for E_{dc}/E_m from Table 3 are plotted as circles in Fig. 9 with the calculated values shown as the solid curve. The data are plotted a little differently than that used in the presentation of the characteristics of Fig. 6. The quantity ωCR is used as the variable in Fig. 9 while $\omega^2 LC$ has the fixed value of 2.0. The zero is not suppressed so that a better idea of the agreement of experimental and calculated results may be obtained. One experimental point, that for $\omega CR = 1.929$, was not used because it lay very close

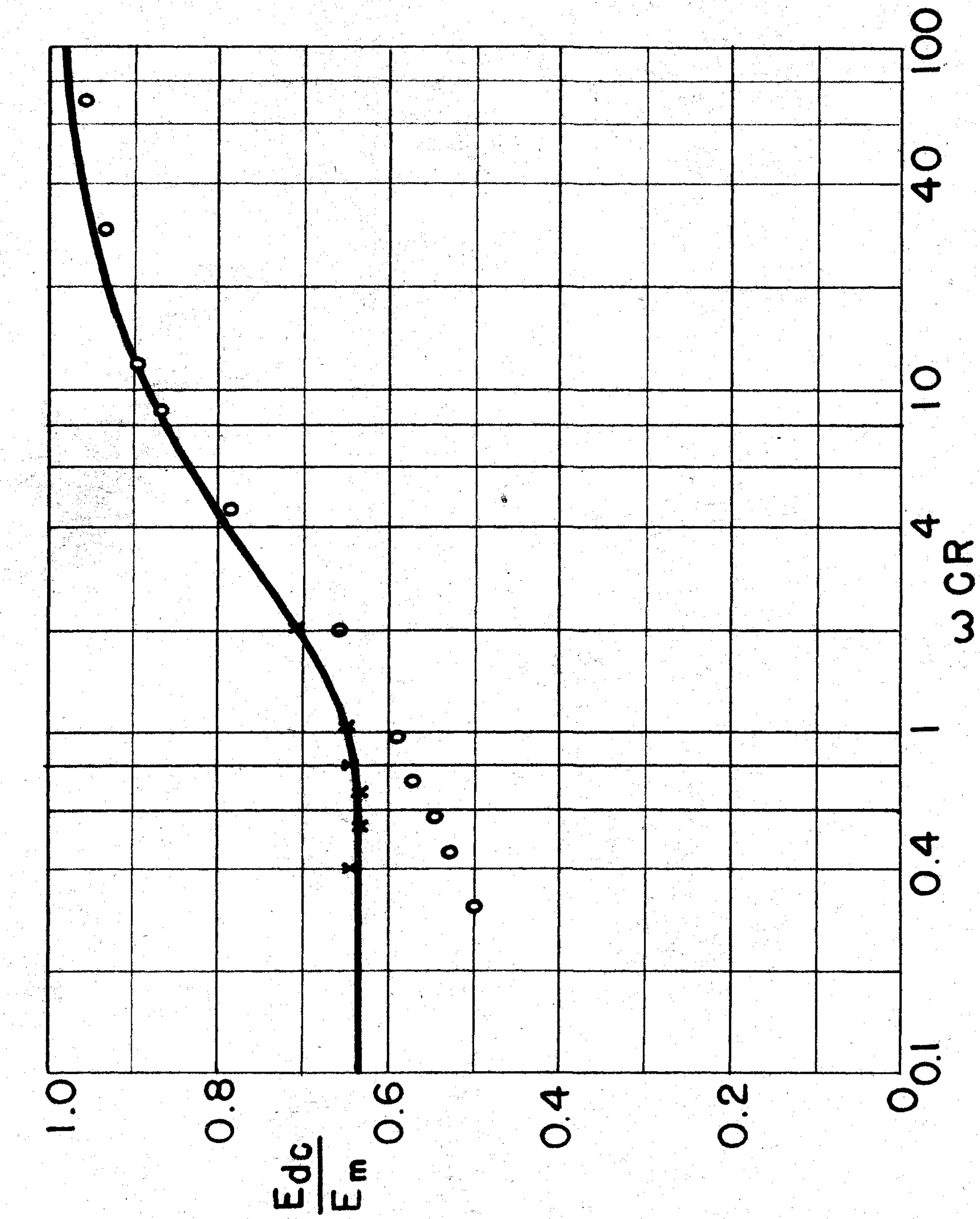

Fig. 9. Average Output Voltage for $\omega^2LC = 2.0$. The solid curve is calculated, the circles are uncompensated experimental points, and the crosses are compensated experimental points.

to another of the experimental points. The agreement between calculated and experimental values is very good for values of ωCR above 3.0. The slight discrepancies between calculated and experimental values for high values of ωCR in the neighborhood of 20 and higher are probably caused by the tube drop and by the slow rate of rise of tube current when conduction starts initially. The time of rise was probably of the order of 100 microseconds in this test circuit, and the rate of rise was limited probably by the size of the input capacitance and the size of the leakage reactance of the transformer. For values of ωCR below 3.0, the difference between experimental and calculated values increases very rapidly. A much better agreement between the calculated and experimental values is obtained if the tube and inductor resistance is taken into account. If the transformer has appreciable resistance, the resistance of one half of the secondary should be used also. The direct-current resistance of the filter inductor was 33.18 ohms and of the tubes was 91.7 ohms, giving a total combined resistance of 124.9 ohms. This combined resistance should be added to the resistance as calculated by dividing E_{dc} by I to obtain the total load resistance. This is equivalent to considering the load resistance increased by the amount of tube and inductor resistance. This new total load resistance when multiplied by ωC will give a new value of ωCR. Both the old values of ωCR from Table 3 and the new compensated values of ωCR are displayed in Table 4. No very great change in ωCR occurs except for the values below 3.0. If the new total load resistance is multiplied by the current I, both a new value of E_{dc} and a new ratio E_{dc}/E_m result. This compensated ratio E_{dc}/E_m is given in Table 4 and is plotted as crosses in Fig. 9. Excellent agreement between the calculated and the compensated experimental results is evident in the figure. Since the compensated points differ very little from the uncompensated points for ωCR larger than 3.0, none of the compensated points are shown for ωCR larger than 3.0.

H. The Results for the Per Cent Ripple

The uncompensated experimental data for the per cent ripple r from Table 3 are plotted as circles in Fig. 10 with the calculated values shown as a solid curve. The curve has the same shape and is plotted in the same manner as the ripple characteristics of Fig. 7 except that the vertical scale is linear not logarithmic. The agreement between the calculated and experimental points is very good for the region of $\omega CR \geq 2.0$. Below this value of ωCR the discrepancy increases rapidly and is largely caused by

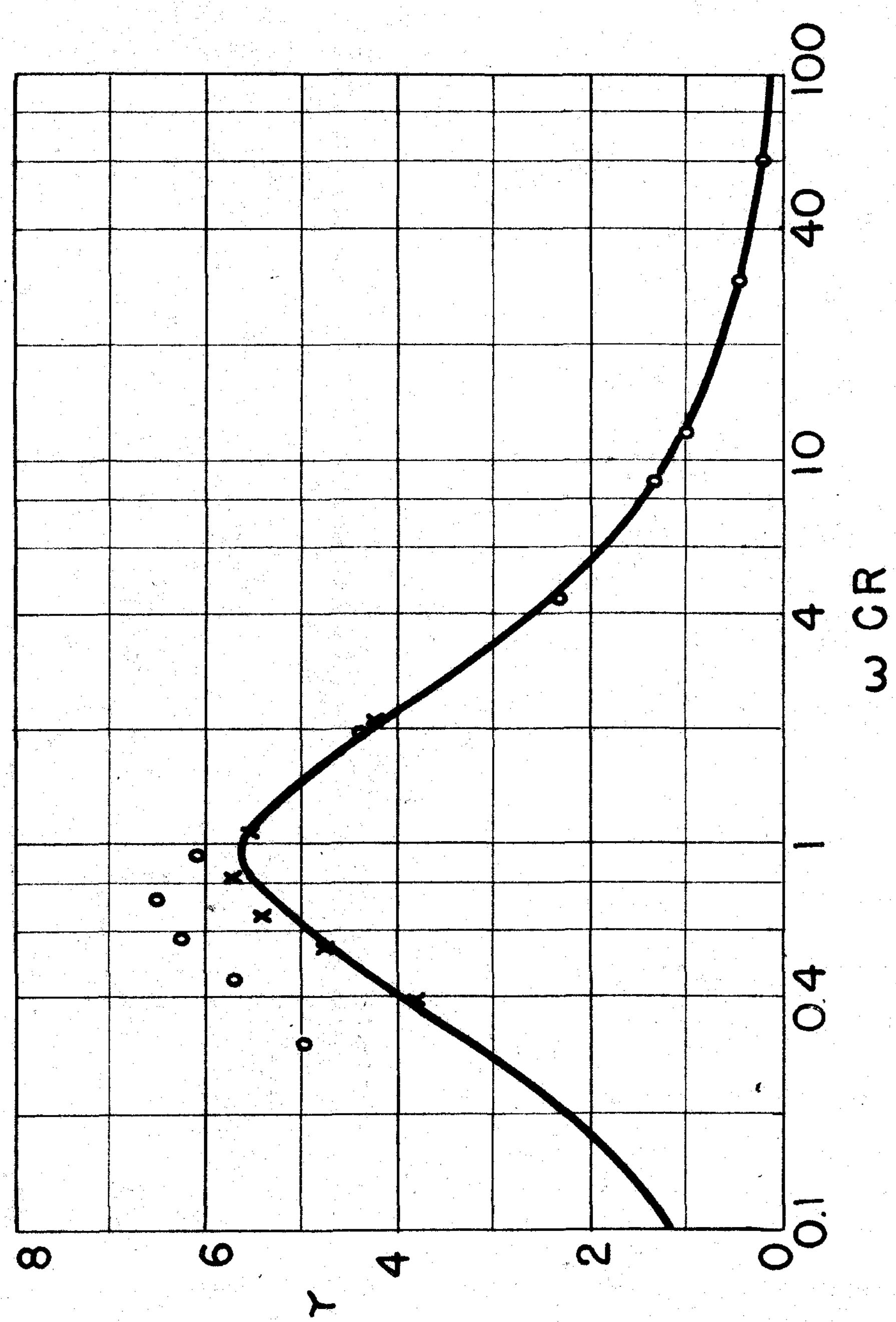

Fig. 10. Per Cent Ripple Voltage for $\omega^2 LC = 2.0$. The Solid curve is calculated, the circles are uncompensated experimental points, and the crosses are compensated experimental points.

the increasing voltage drop in the tubes and the filter inductor. It should be remarked that this voltage drop occurring in the tube and inductor is important only for the direct voltage not for the alternating components of the ripple voltage. For example, the magnitude of the impedance

TABLE 4

Values of Table 3 Compensated for Tube Resistance

ω CR from Table 3	compensated ω CR	compensated E_{dc}/E_m	compensated r in per cent
70.45	70.50	0.9578	0.1931
29.40	29.44	0.9390	0.4472
12.06	12.14	0.9045	0.982
8.885	8.965	0.8800	1.329
4.455	4.543	0.8055	2.273
1.953	2.046	0.7085	4.240
0.731	0.8215	0.6470	5.787
1.929	2.021	0.7040	4.415
0.960	1.052	0.6490	5.560
0.5793	0.6707	0.6360	5.430
0.4492	0.5406	0.6365	4.790
0.3097	0.4010	0.6460	3.856

of the filter inductor has a large effect on the ripple voltage appearing in the load, but this magnitude is changed very little by considering the effect of the presence or absence of resistance in the inductor. All of the ripple voltage should be considered as appearing across the original load resistance and none across the tube or filter inductor resistances. If the measured ripple voltage of Table 3 is divided by the compensated average output voltage E_{dc} and multiplied by 100, the compensated per cent ripple r results and is given in Table 4. These values of the compensated ripple r are plotted as crosses in Fig. 10 and indicate much better agreement between the calculated and experimental values. The small discrepancies still remaining are probably caused by the fact that the tube angle of conduction γ is altered by the effect of the tube and inductor resistance, and no method of correcting for this error is known outside of an extension of the present analysis taking these factors into account.

VI. USE OF CHARACTERISTICS IN DESIGN

Probably the most common problem in rectifier circuit design is one in which the maximum allowable per cent ripple across the load, the load current and voltage, and the input voltage and frequency are known, and a rectifier circuit and filter is needed to satisfy these requirements. The choice as to which rectifier and filter circuit to use, is dictated by a number of factors among which are the technical requirements enumerated above, the cost of the parts, the parts available and the parts that can be easily procured, the size and weight of the completed apparatus, the atmospheric conditions under which it must operate, such as temperature, pressure, and humidity, the reliability in service of the finished article, the type of service such as continuous, intermittent, or pulsed, and the voltage regulation.

The single-phase, full-wave circuit with a condenser-input filter considered in this thesis has as its chief rival in use, the single-phase, full-wave circuit with a choke-input filter. The main advantages of the circuit with the condenser-input filter are its higher voltage and less ripple, while its disadvantages are the higher voltage regulation and larger peak tube currents. The condenser-input type of rectifier circuit is, because of these facts, an excellent power supply for the smaller devices using electronic tubes and requiring a fixed amount of current such as radios, oscilloscopes, amplifiers, and oscillators.

Assuming that the maximum ripple, the load current and voltage, and input voltage and frequency are known, the first step is to select the size of the filter condenser that would be most suitable considering the cost, ease in procuring, and amount of filtering. Obtain the magnitude of ωCR by finding the product of 2π, the input frequency, the capacitance of the filter condenser, and the load resistance which is the load voltage divided by the load current. From Fig. 7 which shows the ripple characteristics select the value of $\omega^2 LC$ that for the calculated value of ωCR will give the maximum permissible ripple. Calculate the amount of inductance L necessary in the filter from this value of $\omega^2 LC$, and see if an inductance of this amount or slightly higher is available. If one can be gotten and if it will carry the load current, this inductor should be used. If it is impossible to find an inductor satisfying these requirements, a new value of the filter capacitance should be selected, and the whole procedure carried through again. When the final values of capacitance and inductance have been decided upon, the quantities

ωCR and $\omega^2 LC$ may be calculated. Then by the use of Fig. 6 which gives the variation of E_{dc}/E_m, the value of E_{dc}/E_m may be obtained, and from the known value of the load voltage the value of E_m may be calculated. This will determine the voltage rating of the secondary of the transformer, and the load current will determine the current rating. Here again an adjustment may be necessary if the ratings of the available transformers are not close enough to the values required. To select the tube, the three quantities, peak tube current, average tube current, and peak inverse tube voltage are necessary. The average tube current is one half the load current I, the peak tube current may be obtained from (11), and the peak inverse tube voltage is $2E_m$. A tube should be selected that has all three of these ratings higher than those required by the circuit.

The tube resistance should be calculated from its characteristics by dividing the voltage across it by the current through it at the point where the current through it is equal to one half the load current. The sum of the tube resistance, the direct current resistance of the filter inductor, and the direct current resistance of one half of the transformer secondary should be added together. If this sum is an appreciable part of the load resistance, say three to five per cent or more, both ωCR and E_{dc} should be increased by this same per cent, and the whole design procedure repeated. It should be remembered, however, that the increase in E_{dc} does not appear at the load but is the average voltage drop lost in the tubes and the filter inductor.

VII. DISCUSSION

There are several extensions of this analysis that could, it is felt, be profitably investigated further. The first is the behavior of the circuit for values of a between 0.6 and 0.0 including the region in which filter resonance occurs. This might necessitate an extensive revision of the mathematical procedure employed in this thesis. The second extension is that of the half-wave rectifier with condenser-input filter. The half-wave rectifier is becoming more and more popular in small radio receiving sets, and an analysis suitable for design purposes in needed. This analysis probably could be done by the method of Appendix B with π in the expressions for H_1, H_2, and H_3 replaced by 2π. The third one is that of taking account of the transformer reactance. Probably the first step to be done here is an extension of the case considered by Freeman[1], i.e., the case of $a \to \infty$, taking into account both the transformer reactance and the tube and transformer resistance and making the analysis general enough so that it covers the full range of the other variables and so that it is not limited by one particular value such as one particular frequency. The next step would be the removal of the restriction of $a \to \infty$. This would undoubtedly prove to be a difficult problem to solve, and it is quite questionable as to whether the labor to obtain such a solution could be justified.

VIII. SUMMARY

In the last fifteen years a number of analyses of the full-wave rectifier circuit with a condenser-input filter have been made, and several of these analyses have presented characteristic curves which could be used in the design of the circuits. These analyses used rather severe assumptions in most cases, and the characteristics were applicable only over a small range of parameters. The present investigation was started with the idea of using more liberal assumptions and producing characteristics suitable for design work that could be used from open-circuited to short-circuited loads and with all values of filter inductances in practical use.

In the solution of the circuit equations, the usual method of differential equations was tried and abandoned, and a new steady-state operational calculus developed for this type of problem was tried and was successful in solving the circuit equations. This solution was made for various values of the parameters and was extended to the cases where one or more of the parameters became zero or infinitely large.

The boundary between the cut-off and the non-cut-off types of operation was obtained, and the solution of the non-cut-off case was made. The point of filter resonance was approached very closely, but difficulties with the solution prevented further investigation of the resonance condition.

Characteristic curves and equations suitable for use in design and covering the range from open-circuited to short-circuited loads and from very large filter inductances to such small ones that filter resonance almost occurs, were constructed. A method of using these characteristics in the actual design of rectifier and filter circuits was also presented.

Experimental results of the operation of an actual circuit were presented and found to agree very closely with the calculated results particularly when correction is made for the resistance of the tube and filter inductance. The correction can also be applied in the design work. Several possible future extensions to the present analysis and characteristics were also suggested.

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XI. VITA

Author: Donald Long Waidelich

Father's name: John Albert Waidelich

Mother's maiden name: Maisie Hamilton Long

Date and place of birth: May 3, 1915, Allentown, Pennsylvania

High School attended: Allentown High School, Allentown, Penna.

Universities attended: Lehigh University, Bethlehem, Penna.
University of Missouri, Columbia, Missouri
Iowa State College, Ames, Iowa

Degrees: Bachelor of Science in Electrical Engineering, Lehigh University, 1936
Master of Science, Lehigh University, 1938

Field of Specialization: Electrical engineering, electronics and communication

Positions held:

1. Teaching Assistant, Lehigh University, Bethlehem, Pennsylvania, 1936-1938.
2. Member of Technical Staff, Bell Telephone Laboratories, New York, N.Y., Summer 1937.
3. Instructor and Assistant Professor of Electrical Engineering, University of Missouri, Columbia, Missouri, 1938-1944.
4. Member of Research Staff, Westinghouse Electric Corporation, Bloomfield, N.J., Summer 1941.
5. Electrical Engineer, Naval Ordnance Laboratory, Washington, D.C., 1944-1945.

XII. APPENDIX A-NOMENCLATURE

a = $\omega^2 LC$.

A_1 = a quantity defined by (B-7).

A_2 = the reading in amperes of the meter to measure direct current in Fig. 8.

b = ωCR.

C = the capacitance in farads of either the first or second filter condenser.

C_1 = the first filter condenser of Fig. 2.

C_2 = the second filter condenser of Fig. 2.

C_3, C_4 = the capacitance in farads of the decade capacitance boxes of Fig. 8.

C_5, C_6 = by-pass, blocking condensers of Fig. 8.

D = a determinant defined by (B-25).

D_0, D_1, D_2, D_3, D_4 = quantities defined by (B-39).

e = the instantaneous value in volts of the applied sinusoidal voltage.

$e_1(t)$ = $q_1(t)/C$ = the instantaneous voltage across the first condenser.

E = $\int_{\alpha/\omega}^{\beta/\omega} \epsilon^{-pt} e\,dt$.

E_c = a quantity defined by (B-36) and (B-37).

E_m = the maximum value in volts of the applied sinusoidal voltage.

E_s = a quantity defined by (B-36) and (B-37).

E_{dc} = IR = the average voltage across the load resistor R.

f = the frequency in cycles per second of the applied sinusoidal voltage.

$f(t)$ = function of t with the period T.

$F(p)$ = a function of p.

F_1, F_2, F_3, F_4 = quantities defined by (B-10).

$$g = \sqrt{\frac{\frac{1}{b^2}+1}{\frac{1}{b^2}+\left(\frac{1}{a}-1\right)^2}}$$ and g is positive.

$g(t)$ = a function of t with the period T.

G_1, G_2 = quantities defined by (B-20).

G_3 = a quantity defined by (B-31).

H_1, H_2, H_3 = quantities defined by (B-23).

i_m = peak tube current.

$i(t)$ = the instantaneous current in amperes flowing through the load resistor.

$i_L(t)$ = the instantaneous current in amperes flowing through the filter inductor.

$i_T(t)$ = the instantaneous current in amperes flowing through the rectifier tube.

$i_1(t)$ = the instantaneous current in amperes flowing into the first condenser.

$i_2(t)$ = the instantaneous current in amperes flowing into the second condenser.

I = the average current flowing through the load resistor R.

I_{AC} = the alternating current in amperes flowing in the inductor of Fig. 8.

I_{DC} = the direct current in amperes flowing in the inductor of Fig. 8.

I_1 = $\int_{\alpha/\omega}^{\beta/\omega} \epsilon^{-pt} i_1 dt$ with similar expressions for the other instantaneous currents.

$K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9$ = quantities defined by (B-27).

L = the inductance in henries of the filter inductor.

$$m_1 = \frac{1}{2b} + \sqrt{\left(\frac{1}{2b}\right)^2 - \frac{1}{a}}$$

$$m_2 = \frac{1}{2b} - \sqrt{\left(\frac{1}{2b}\right)^2 - \frac{1}{a}}$$

m_{1C} = imaginary part of m_1, if m_1 is complex.

m_{1R} = real part of m_1, if m_1 is complex.

M_1, M_2, M_3 = quantities defined by (B-29).

n = a positive or negative integer or zero.

n_1, n_2, n_3 = quantities defined by (B-10).

n_4 = n_1.

n_5 = n_2.

n_{2C} = imaginary part of n_2.

n_{2R} = real part of n_2.

N_1, N_2, N_3 = quantities defined by (B-31).

p = a complex number.

P_T = $2i_m/I$ = ratio of peak to average tube current.

$q_1(t)$ = the instantaneous charge in coulombs on the first filter condenser.

$q_2(t)$ = the instantaneous charge in coulombs on the second filter condenser.

r = per cent ripple voltage across the load resistor R.

R = the resistance in ohms of the load resistor.

R_L = the apparent resistance in ohms of the filter inductor.

R_1, R_2, R_4 = the resistance in ohms of the decade resistance boxes of Fig. 8.

S = direct trnsform.

S^{-1} = inverse transform.

t = time in seconds.

t_1, t_2 = specific values of the time t.

T = period of voltage or currents in seconds.

T_1, T_2 = the tubes of Fig. 2.

U_1, U_2 = quantities defined by (D-2).

v = a positive integer.

V_1 = the reading in volts of the meter to measure alternating voltage in Fig. 8.

W, W_1, W_2, W_3 = paths of integration for the inverse transform.

x_1, x_2, x_3, x_4, x_5, x_6 = quantities defined by (B-14).

y = a quantity defined by (D-3).

α = the angle in radians at which the tube T_1 starts conducting.

β = the angle in radians at which the tube T_1 stops conducting.

β_1 = an angle defined by (B-28).

β_2 = an angle defined by (B-30).

$\gamma = \beta - \alpha$ = the angle in radians of the time the tube T_1 is conducting.

$\lambda = \tan^{-1}\left[\frac{a}{b(1-a)}\right] - \tan^{-1} b$ and $-\pi/2 \leq \lambda \leq +\pi/2$.

$\omega = 2\pi f = \pi/T$.

$\omega_1 = (2\pi/T)$.

XIII. APPENDIX B-SOLUTION OF THE CIRCUIT EQUATIONS FOR THE GENERAL CASE

A. The Equations of the Equivalent Circuits

The equivalent circuit shown in Fig. 3(a) applies from $t = (\alpha/\omega)$ to $t = (\beta/\omega)$, i.e., the time that the tube T_1 is conducting. The circuit of Fig. 3(b) applies from $t = 0$ to $t = (\alpha/\omega)$ and from $t = (\beta/\omega)$ to $t = T = (\pi/\omega)$, i.e., when the tube T_1 is not conducting. Hence i_T must be zero from $t = 0$ to $t = (\alpha/\omega)$ and from $t = (\beta/\omega)$ to $t = T$, but i_T will have a value dictated by the circuit and the applied voltage from $t = (\alpha/\omega)$ to $t = (\beta/\omega)$. The tube current i_T will be solved for by using Fig. 3(a) first. The circuit equations are:

$$\left.\begin{aligned} e &= q_1/C \\ e &= L(di_L/dt) + Ri \\ Ri &= q_2/C \\ i_T &= i_1 + i_L \\ i_L &= i + i_2 \, . \end{aligned}\right\} \quad \text{(B-1)}$$

Let $E = \int_{\alpha/\omega}^{\beta/\omega} \epsilon^{-pt} e\,dt$

and $I_T = \int_{\alpha/\omega}^{\beta/\omega} \epsilon^{-pt} i_T \, dt$ with similar expressions for the other currents. The equations of (B-1) are multiplied by ϵ^{-pt} and by the aid of equations (8) and (9) integrated with respect to t from α/ω to β/ω to obtain:

$$\left.\begin{aligned}
E &= (1/pC)\left[q_1(\alpha/\omega)\epsilon^{-p\alpha/\omega} - q_2(\beta/\omega)\epsilon^{-p\beta/\omega}\right] + (1/pC)I_1 \\
E &= L\left[i_L(\beta/\omega)\epsilon^{-p\beta/\omega} - i_L(\alpha/\omega)\epsilon^{-p\alpha/\omega}\right] + pLI_L + RI \\
RI &= (1/pC)\left[q_2(\alpha/\omega)\epsilon^{-p\alpha/\omega} - q_2(\beta/\omega)\epsilon^{-p\beta/\omega}\right] + (1/pC)I_2 \\
I_T &= I_1 + I_L \\
I_L &= I + I_2.
\end{aligned}\right\} \quad (B\text{-}2)$$

The equations (B-2) are then solved for I_T:

$$\begin{aligned}
I_T = {} & pCE + \frac{E}{pL + \frac{R(1/pC)}{R + 1/pC}} \\
& - \frac{L\left[i_L(\beta/\omega)\epsilon^{-p\beta/\omega} - i_L(\alpha/\omega)\epsilon^{-p\alpha/\omega}\right]}{pL + \frac{R(1/pC)}{R + 1/pC}} \\
& + \frac{\frac{R(1/pC)}{R + 1/pC}}{pL + \frac{R(1/pC)}{R + 1/pC}}\left[q_2(\beta/\omega)\epsilon^{-p\beta/\omega} - q_2(\alpha/\omega)\epsilon^{-p\alpha/\omega}\right] \\
& + \left[q_1(\beta/\omega)\epsilon^{-p\beta/\omega} - q_1(\alpha/\omega)\epsilon^{-p\alpha/\omega}\right].
\end{aligned} \quad (B\text{-}3)$$

Since i_T is zero in the intervals $0 \leq t < \alpha/\omega$ and $\beta/\omega \leq t \leq T$, the steady-state direct transform of i_T must be equal to I_T as given in (B-3).

Thus $S(i_T) = I_T$. (B-4)

Furthermore:

$$E = \int_{\alpha/\omega}^{\beta/\omega} \epsilon^{-pt} E_m \sin\omega t \, dt$$

$$= \frac{E_m}{p^2+\omega^2}\left[\epsilon^{-p\alpha/\omega}(p\sin\alpha + \omega\cos\alpha) - \epsilon^{-p\beta/\omega}(p\sin\beta + \omega\cos\beta)\right].$$

(B-5)

The expression for $S(i_T)$ as given in (B-3) contains five terms, the first of which represents the current flowing through the first condenser and the second, the current flowing through the inductor. The third, fourth, and fifth terms represent the effect of the condenser charges and inductor current at the instants the tube starts and stops conducting. These will be treated as unknown constants and will be solved for later on.

One particular term of $S(i_T)$ in (B-3) will be evaluated by the use of the inverse transform as given in (2). This will be the second term, and the others may be evaluated in a similar manner.

$$\frac{E}{pL+\frac{R(1/pC)}{R+1/pC}}$$

$$= \frac{E_m}{L}\frac{(p+1/CR)\left[\epsilon^{-p\alpha/\omega}(p\sin\alpha+\omega\cos\alpha)-\epsilon^{-p\beta/\omega}(p\sin\beta+\omega\cos\beta)\right]}{(p+j\omega)(p-j\omega)(p+\omega m_1)(p+\omega m_2)}$$

where $m_1 = \frac{1}{2b}+\sqrt{\left(\frac{1}{2b}\right)^2-\frac{1}{a}}$ and $m_2 = \frac{1}{2b}-\sqrt{\left(\frac{1}{2b}\right)^2-\frac{1}{a}}$.

The second term of i_T is

$$\frac{1}{2\pi j}\int_{W_1}\epsilon^{pt}\frac{E_m}{L}\frac{(p+1/CR)\left[\epsilon^{-p\alpha/\omega}(p\sin\alpha+\omega\cos\alpha)-\epsilon^{-p\beta/\omega}(p\sin\beta+\omega\cos\beta)\right]}{(p+j\omega)(p-j\omega)(p+\omega m_1)(p+\omega m_2)}dp$$

$$-\frac{1}{2\pi j}\int_{W_3}\frac{\epsilon^{pt}}{1-\epsilon^{-pT}}\frac{E_m}{L}\frac{(p+1/CR)\left[\epsilon^{-p\alpha/\omega}(p\sin\alpha+\omega\cos\alpha)-\epsilon^{-p\beta/\omega}(p\sin\beta+\omega\cos\beta)\right]}{(p+j\omega)(p-j\omega)(p+\omega m_1)(p+\omega m_2)}dp.$$

(B-6)

Both integrals of (B-6) have the four poles, $-j\omega$, $+j\omega$, $-\omega m_1$, and $-\omega m_2$, and the residues at all of these poles must be summed.

When $0 \leq t < \alpha/\omega$, the second term of i_T is

$$-\frac{E_m}{\omega L}\,\frac{\epsilon^{-m_1\omega t}\,m_2\left[\epsilon^{m_1\alpha}(-m_1\sin\alpha+\cos\alpha)-\epsilon^{m_1\beta}(-m_1\sin\beta+\cos\beta)\right]}{(1-\epsilon^{m_1\pi})(m_1^2+1)(m_2-m_1)}$$

$$-\frac{E_m}{\omega L}\,\frac{\epsilon^{-m_2\omega t}\,m_1\left[\epsilon^{m_2\alpha}(-m_2\sin\alpha+\cos\alpha)-\epsilon^{m_2\beta}(-m_2\sin\beta+\cos\beta)\right]}{(1-\epsilon^{m_2\pi})(m_2^2+1)(m_1-m_2)}.$$

(B-7)

Let the quantity (B-7) be denoted by A_1.

When $\alpha/\omega \leq t \leq \beta/\omega$, the second term of i_T is

$$\frac{E_m}{\omega L}\left[\frac{\epsilon^{j\omega t}(j+\frac{1}{\omega CR})}{2j(j+m_1)(j+m_2)}-\frac{\epsilon^{-j\omega t}(-j+\frac{1}{\omega CR})}{2j(-j+m_1)(-j+m_2)}\right]+A_1$$

$$+\frac{E_m}{\omega L}\left\{\frac{\epsilon^{-m_1\omega t}\,m_2\left[\epsilon^{m_1\alpha}(-m_1\sin\alpha+\cos\alpha)\right]}{(m_1^2+1)(m_2-m_1)}+\frac{\epsilon^{-m_2\omega t}\,m_1\left[\epsilon^{m_2\alpha}(-m_2\sin\alpha+\cos\alpha)\right]}{(m_2^2+1)(m_1-m_2)}\right\}.$$

When $\beta/\omega \leq t \leq T$, the second term of i_T is the same as (B-7) except that the time origin is shifted T seconds forward.

B. The Eight Equations

During the interval $0 \leq t < \alpha/\omega$, i_T should be zero, and this can be fulfilled if the contribution (B-7) from the second term of i_T plus those from the other terms of $S(i_T)$ are put equal to zero. The coefficients of $\epsilon^{-m_1\omega t}$ and of $\epsilon^{-m_2\omega t}$ will each become a separate equation. These are the first two equations of the eight that will have to be used to determine the eight unknowns α, β, $q_1(\alpha/\omega)$, $q_1(\beta/\omega)$, $q_2(\alpha/\omega)$, $q_2(\beta/\omega)$, $i_L(\alpha/\omega)$, $i_L(\beta/\omega)$. The two equations are:

$$\left.\begin{aligned}
&\frac{E_m}{\omega L}\left(\frac{m_2}{m_1^2+1}\right)\left[\epsilon^{m_1\alpha}(-m_1\sin\alpha+\cos\alpha)-\epsilon^{m_1\beta}(-m_1\sin\beta+\cos\beta)\right]\\
&+m_2\left[i_L(\alpha/\omega)\epsilon^{m_1\alpha}-i_L(\beta/\omega)\epsilon^{m_1\beta}\right]+\omega m_1 m_2\left[q_2(\beta/\omega)\epsilon^{m_1\beta}-q_2(\alpha/\omega)\epsilon^{m_1\alpha}\right]=0\\
&\frac{E_m}{\omega L}\left(\frac{m_1}{m_2^2+1}\right)\left[\epsilon^{m_2\alpha}(-m_2\sin\alpha+\cos\alpha)-\epsilon^{m_2\beta}(-m_2\sin\beta+\cos\beta)\right]\\
&+m_1\left[i_L(\alpha/\omega)\epsilon^{m_2\alpha}-i_L(\beta/\omega)\epsilon^{m_2\beta}\right]+\omega m_1 m_2\left[q_2(\beta/\omega)\epsilon^{m_2\beta}-q_2(\alpha/\omega)\epsilon^{m_2\alpha}\right]=0.
\end{aligned}\right\}\text{(B-8)}$$

In a similar manner by the use of Fig. 3, the direct transform of i_1 may be found. The condition that in the interval, $\alpha/\omega \leq t \leq \beta/\omega$, i_1 must equal $\omega CE_m \cos\omega t$ leads to three equations which involve the eight unknowns mentioned before. These three equations are:

$$a\left[i_L(\alpha/\omega)\epsilon^{n_v(\alpha+\pi)} - i_L(\beta/\omega)\epsilon^{n_v\beta}\right] + \left(\frac{\omega}{\frac{1}{b}-n_v}\right)\left[q_2(\beta/\omega)\epsilon^{n_v\beta} - q_2(\alpha/\omega)\epsilon^{n_v(\alpha+\pi)}\right]$$

$$+\left(\frac{\omega}{n_v}\right)\left[q_1(\beta/\omega)\epsilon^{n_v\beta} - q_1(\alpha/\omega)\epsilon^{n_v(\alpha+\pi)}\right] = 0$$

(B-9)

where a different equation results for each value of $v = 1, 2, 3$. The three values of n_v are obtained by the following method:

1. $F_1 = \sqrt[3]{F_3 + F_4}$

   $F_2 = \sqrt[3]{F_3 - F_4}$

where $F_3 = \dfrac{1}{6ab} + \dfrac{1}{27b^3}$

$$F_4 = \sqrt{\frac{1}{108a}\left(\frac{32}{a^2} - \frac{13}{ab^2} + \frac{4}{b^4}\right)} \qquad \text{(B-10)}$$

2. $n_1 = \dfrac{1}{3b} + F_1 + F_2$

   $n_2 = \dfrac{1}{3b} - \dfrac{1}{2}(F_1 + F_2) + j\,\dfrac{\sqrt{3}}{2}(F_1 - F_2)$

   $n_3 = \dfrac{1}{3b} - \dfrac{1}{2}(F_1 + F_2) - j\,\dfrac{\sqrt{3}}{2}(F_1 - F_2).$

The starting and stopping properties of the tubes will be used next to obtain the last three equations necessary to solve for the eight unknowns mentioned previously. The tubes are assumed to start conducting when the voltage across them becomes zero, which, in effect, means that the voltage across one half of the secondary of the transformer in Fig. 2 becomes equal to the voltage of the first condenser. Expressed mathematically, this condition is

$$E_m \sin\alpha = q_1(\alpha/\omega)/C,$$

or $$q_1(\alpha/\omega) = CE_m \sin\alpha\,. \qquad \text{(B-11)}$$

The tubes stop conducting when the current flowing through them becomes zero or $i_T(\beta/\omega) = 0$. From Fig. 3, this is equivalent to saying that $i_1(\beta/\omega) = -i_L(\beta/\omega)$

or $$i_L(\beta/\omega) = -\omega CE_m \cos\beta\,. \qquad \text{(B-12)}$$

Since at this instant the voltage of the first condenser must be equal to the voltage across one half the secondary of the transformer,

$$E_m \sin\beta = q_1(\beta/\omega)/C,$$

or $$q_1(\beta/\omega) = CE_m \sin\beta\,. \qquad \text{(B-13)}$$

C. The Solution of the Eight Equations

The eight equations of (B-8), (B-9), (B-11), (B-12), and (B-13) will be solved for the eight unknowns α, β, $q_1(\alpha/\omega)$, $q_1(\beta/\omega)$, $q_2(\alpha/\omega)$, $q_2(\beta/\omega)$, $i_L(\alpha/\omega)$, and $i_L(\beta/\omega)$. A slight change in the variables is desirable for ease in handling these equations, and this is:

let $\gamma = \beta - \alpha$

$$
\begin{aligned}
x_1 &= (\omega L/E_m)\, i_L(\alpha/\omega) \\
x_2 &= (\omega L/E_m)\, i_L(\beta/\omega) \\
x_3 &= (\omega L/E_m)\, \omega q_2(\alpha/\omega) \\
x_4 &= (\omega L/E_m)\, \omega q_2(\beta/\omega) \\
x_5 &= (\omega L/E_m)\, \omega q_1(\alpha/\omega) \\
x_6 &= (\omega L/E_m)\, \omega q_1(\beta/\omega)
\end{aligned} \tag{B-14}
$$

The two equations (B-8) become for v = 1, 2:

$$
\left(\frac{1}{m_v^2+1}\right)\left[\epsilon^{-m_v\gamma}(-m_v \sin\alpha + \cos\alpha) - (-m_v \sin\beta + \cos\beta)\right]
+\left[\epsilon^{-m_v\gamma} x_1 - x_2\right] + m_v\left[x_4 - \epsilon^{-m_v\gamma} x_3\right] = 0. \tag{B-15}
$$

The three equations (B-9) become for v = 1, 2, 3:

$$
\left[x_2 \epsilon^{-n_v(\pi-\gamma)} - x_1\right] + \frac{1}{a(\frac{1}{b}-n_v)}\left[x_3 - \epsilon^{-n_v(\pi-\gamma)} x_4\right]
+\frac{1}{a n_v}\left[x_5 - \epsilon^{-n_v(\pi-\gamma)} x_6\right] = 0. \tag{B-16}
$$

Equations (B-11), (B-12), and (B-13) become:

$$x_5 = a \sin\alpha \tag{B-17}$$

$$x_2 = -a \cos\beta \tag{B-18}$$

$$x_6 = a \sin\beta \,. \tag{B-19}$$

The next step will be to eliminate the complex numbers occurring in equations (B-15) and (B-16). Two auxiliary real functions of γ that are symmetrical in m_1 and m_2 are defined:

$$G_1 = \epsilon^{-m_1\gamma} + \epsilon^{-m_2\gamma} = 2\epsilon^{-m_{1R}\gamma}\cos m_{1C}\gamma$$

$$G_2 = m_1\epsilon^{-m_1\gamma} + m_2\epsilon^{-m_2\gamma}$$

$$= 2\epsilon^{-m_{1R}\gamma}(m_{1R}\cos m_{1C}\gamma + m_{1C}\sin m_{1C}\gamma)$$

(B-20)

where if m_1 is a complex number, $m_1 = m_{1R} + jm_{1C}$. If the two equations of (B-15) are added together, the resulting equation has no complex terms in it.

$$-G_2\sin\alpha + G_1\cos\alpha + \frac{1}{b}\sin\beta - 2\cos\beta + \left[\frac{G_2}{b} + G_1\left(1-\frac{1}{a}\right)\right]\chi_1$$

$$-\left(\frac{1}{b^2} - \frac{2}{a} + 2\right)\chi_2 + \frac{1}{b}\left(\frac{1}{b^2} - \frac{3}{a} + 1\right)\chi_4 - \left[G_2\left(1 - \frac{1}{a} + \frac{1}{b^2}\right) - \frac{G_1}{ab}\right]\chi_3 = 0.$$

(B-21)

Multiply the first equation of (B-15) by m_1, the second by m_2, and add:

$$-G_1\sin\alpha + a\left(\frac{G_1}{b} - G_2\right)\cos\alpha + 2\sin\beta - \frac{a}{b}\cos\beta + \left[\frac{a}{b}G_1 + (1-a)G_2\right]\chi_1$$

$$-\frac{1}{b}(1+a)\chi_2 + \left(\frac{1}{b^2} - \frac{2}{a} + 2\right)\chi_4 - \left[\frac{G_2}{b} + \left(1 - \frac{1}{a}\right)G_1\right]\chi_3 = 0.$$

(B-22)

Similarly for (B-16), three auxiliary real functions of γ that are symmetrical in n_1, n_2, n_3 are defined:

$$\left.\begin{aligned}
H_1 &= \epsilon^{-n_1(\pi-\gamma)} + \epsilon^{-n_2(\pi-\gamma)} + \epsilon^{-n_3(\pi-\gamma)} \\
&= \epsilon^{-n_1(\pi-\gamma)} + 2\epsilon^{-n_{2R}(\pi-\gamma)} \cos\left[n_{2C}(\pi-\gamma)\right] \\
H_2 &= n_1\epsilon^{-n_1(\pi-\gamma)} + n_2\,\epsilon^{-n_2(\pi-\gamma)} + n_3\,\epsilon^{-n_3(\pi-\gamma)} \\
&= n_1\epsilon^{-n_1(\pi-\gamma)} + 2\epsilon^{-n_{2R}(\pi-\gamma)}\left\{n_{2R}\cos\left[n_{2C}(\pi-\gamma)\right] + n_{2C}\sin\left[n_{2C}(\pi-\gamma)\right]\right\} \\
H_3 &= \frac{1}{n_1}\epsilon^{-n_1(\pi-\gamma)} + \frac{1}{n_2}\epsilon^{-n_2(\pi-\gamma)} + \frac{1}{n_3}\epsilon^{-n_3(\pi-\gamma)} \\
&= \frac{1}{n_1}\epsilon^{-n_1(\pi-\gamma)} + \frac{2\epsilon^{-n_{2R}(\pi-\gamma)}}{n_{2R}^2 + n_{2C}^2}\left\{n_{2R}\cos\left[n_{2C}(\pi-\gamma)\right] - n_{2C}\sin\left[n_{2C}(\pi-\gamma)\right]\right\}
\end{aligned}\right\} \quad \text{(B-23)}$$

where $n_2 = n_{2R} + jn_{2C}$.

The equations (B-16) then become with the aid of (B-17), (B-18), (B-19), and (B-23):

$$\left.\begin{aligned}
-\frac{2a}{b}x_1 + 3x_3 - H_1x_4 &= a^2\left(\frac{H_1}{b} - H_2\right)\cos\beta + a\sin\alpha + a\left(\frac{H_3}{b} - H_1\right)\sin\beta \\
-4x_1 + \frac{1}{b}x_3 - H_2x_4 &= a\left(2H_1 - \frac{H_3}{b}\right)\cos\beta - \frac{2a}{b}\sin\alpha + a\left(\frac{H_1}{b} - H_2\right)\sin\beta \\
a\,x_1 + 2b\,x_3 - H_3x_4 &= a^2\left(\frac{H_3}{b} - H_1\right)\cos\beta - 2a\left(b - \frac{a}{b}\right)\sin\alpha + a\left(aH_2 - \frac{a}{b}H_1 + H_3\right)\sin\beta.
\end{aligned}\right\} \quad \text{(B-24)}$$

The equations (B-24) can now be solved for x_1, x_3, and x_4. The first step will be to define the determinant

$$D = \begin{vmatrix} -\frac{2a}{b} & 3 & -H_1 \\ -4 & \frac{1}{b} & -H_2 \\ a & 2b & -H_3 \end{vmatrix} \tag{B-25}$$

Then

$$\left.\begin{aligned} x_1 &= K_1 \cos\beta + K_2 \sin\alpha + K_3 \sin\beta \\ x_3 &= K_4 \cos\beta + K_5 \sin\alpha + K_6 \sin\beta \\ x_4 &= K_7 \cos\beta + K_8 \sin\alpha + K_9 \sin\beta \end{aligned}\right\} \tag{B-26}$$

where

$$K_1 = \frac{1}{D}\begin{vmatrix} a^2\left(\frac{H_1}{b} - H_2\right) & 3 & -H_1 \\ a\left(2H_1 - \frac{H_3}{b}\right) & \frac{1}{b} & -H_2 \\ a^2\left(\frac{H_3}{b} - H_1\right) & 2b & -H_3 \end{vmatrix}$$

$$K_2 = \frac{1}{D}\begin{vmatrix} a & 3 & -H_1 \\ -\frac{2a}{b} & \frac{1}{b} & -H_2 \\ 2a\left(\frac{a}{b} - b\right) & 2b & -H_3 \end{vmatrix} \tag{B-27}$$

$$K_3 = \frac{1}{D}\begin{vmatrix} a\left(\frac{H_3}{b} - H_1\right) & 3 & -H_1 \\ a\left(\frac{H_1}{b} - H_2\right) & \frac{1}{b} & -H_2 \\ a\left(aH_2 - \frac{a}{b}H_1 + H_3\right) & 2b & -H_3 \end{vmatrix}$$

$$K_4 = \frac{1}{D}\begin{vmatrix} -\frac{2a}{b} & a^2\left(\frac{H_1}{b} - H_2\right) & H_1 \\ -4 & a\left(2H_1 - \frac{H_3}{b}\right) & -H_2 \\ a & a^2\left(\frac{H_3}{b} - H_1\right) & -H_3 \end{vmatrix}$$

$$K_5 = \frac{1}{D}\begin{vmatrix} -\frac{2a}{b} & a & -H_1 \\ -4 & -\frac{2a}{b} & -H_2 \\ a & 2a\left(\frac{a}{b} - b\right) & -H_3 \end{vmatrix}$$

$$K_6 = \frac{1}{D}\begin{vmatrix} -\frac{2a}{b} & a\left(\frac{H_3}{b} - H_1\right) & -H_1 \\ -4 & a\left(\frac{H_1}{b} - H_2\right) & -H_2 \\ a & a(aH_2 - \frac{a}{b} H_1 + H_3) & -H_3 \end{vmatrix}$$

$$K_7 = \frac{1}{D}\begin{vmatrix} -\frac{2a}{b} & 3 & a^2\left(\frac{H_1}{b} - H_2\right) \\ -4 & \frac{1}{b} & a\left(2H_1 - \frac{H_3}{b}\right) \\ a & 2b & a^2\left(\frac{H_3}{b} - H_1\right) \end{vmatrix}$$

$$K_8 = \frac{1}{D}\begin{vmatrix} -\frac{2a}{b} & 3 & a \\ -4 & \frac{1}{b} & -\frac{2a}{b} \\ a & 2b & 2a\left(\frac{a}{b} - b\right) \end{vmatrix}$$

$$K_9 = \frac{1}{D}\begin{vmatrix} -\frac{2a}{b} & 3 & a\left(\frac{H_3}{b} - H_1\right) \\ -4 & \frac{1}{b} & a\left(\frac{H_1}{b} - H_2\right) \\ a & 2b & a\left(aH_2 - \frac{a}{b}H_1 + H_3\right) \end{vmatrix}$$

With the use of the substitution $\alpha = \beta - \gamma$ and putting (B-26) and (B-18) into (B-21), the following equation results:

$$\tan\beta_1 = -\frac{-M_2 \sin\gamma + G_1 \cos\gamma + M_1}{G_1 \sin\gamma + M_2 \cos\gamma + M_3} \tag{B-28}$$

where

$$\left.\begin{aligned} M_1 &= \frac{a}{b^2} + 2a - 4 + K_1\left[\frac{G_2}{b} + G_1\left(1 - \frac{1}{a}\right)\right] + K_4\left[\frac{G_1}{ab} - G_2\left(1 - \frac{1}{a} + \frac{1}{b^2}\right)\right] + K_7\left[\frac{1}{b}\left(\frac{1}{b^2} - \frac{3}{a} + 1\right)\right] \\ M_2 &= -G_2 + K_2\left[\frac{G_2}{b} + G_1\left(1 - \frac{1}{a}\right)\right] + K_5\left[\frac{G_1}{ab} - G_2\left(1 - \frac{1}{a} + \frac{1}{b^2}\right)\right] + K_8\left[\frac{1}{b}\left(\frac{1}{b^2} - \frac{3}{a} + 1\right)\right] \\ M_3 &= \frac{1}{b} + K_3\left[\frac{G_2}{b} + G_1\left(1 - \frac{1}{a}\right)\right] + K_6\left[\frac{G_1}{ab} - G_2\left(1 - \frac{1}{a} + \frac{1}{b^2}\right)\right] + K_9\left[\frac{1}{b}\left(\frac{1}{b^2} - \frac{3}{a} + 1\right)\right]. \end{aligned}\right\} \tag{B-29}$$

A similar procedure applied to (B-22) results in

$$\tan\beta_2 = -\frac{-N_2 \sin\gamma + G_3 \cos\gamma + N_1}{G_3 \sin\gamma + N_2 \cos\gamma + N_3} \tag{B-30}$$

where

$$\left.\begin{aligned}
N_1 &= \frac{a^2}{b} + K_1\left[\frac{a}{b}G_1 + (1-a)G_2\right] - K_4\left[\frac{G_2}{b} + G_1\left(1-\frac{1}{a}\right)\right] + K_7\left[\frac{1}{b^2} - \frac{2}{a} + 2\right] \\
N_2 &= -G_1 + K_2\left[\frac{a}{b}G_1 + (1-a)G_2\right] - K_5\left[\frac{G_2}{b} + G_1\left(1-\frac{1}{a}\right)\right] + K_8\left[\frac{1}{b^2} - \frac{2}{a} + 2\right] \\
N_3 &= 2 + K_3\left[\frac{a}{b}G_1 + (1-a)G_2\right] - K_6\left[\frac{G_2}{b} + G_1\left(1-\frac{1}{a}\right)\right] + K_9\left[\frac{1}{b^2} - \frac{2}{a} + 2\right] \\
G_3 &= a\left(\frac{1}{b}G_1 - G_2\right).
\end{aligned}\right\} \quad \text{(B-31)}$$

D. The Method of Computation

The equations are now suitable for numerical computation of individual points, and the method of solution for a particular point is essentially the following:

1. Select the value of a and of b for the point to be calculated.

2. Calculate $m_1 = \frac{1}{2b} + \sqrt{\left(\frac{1}{2b}\right)^2 - \frac{1}{a}}$ and $m_2 = \frac{1}{2b} - \sqrt{\left(\frac{1}{2b}\right)^2 - \frac{1}{a}}$.

3. Calculate n_1 and n_2 by means of (B-10).

4. Estimate the value of γ, i.e., the angle during which one tube conducts. Values of γ obtained experimentally were found very helpful in this connection.

5. Calculate H_1, H_2, H_3 by the use of (B-23).

6. Calculate D from (B-25).

7. Calculate K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9 by means of (B-27).

8. Calculate G_1, G_2, G_3 from (B-20) and (B-31).

9. Calculate M_1, M_2, M_3 by the use of (B-29).

10. Calculate N_1, N_2, N_3 by means of (B-31).

11. Calculate β_1 and β_2 by the use of (B-28) and (B-30).

12. If the two angles β_1 and β_2 as calculated in part 11. were the same, the value of γ as estimated in part 4. was correct. For almost all estimates of γ the two angles β_1 and β_2 will not be the same. Then a new estimate of γ must be made, and parts 5. to 12. inclusive carried through again. This procedure should be repeated until β_1 and β_2 are equal considering the accuracy of the tables and computing aids used.

The sequence of the above steps should be preserved since, for example, part 7. requires the results of part 6.

To illustrate the use of the above method as applied to the calculation of a particular point, the following was selected from the calculations made for this thesis:

1. $a = 0.6$, $b = 5.0$

2. $m_1 = 0.1000 + j1.287$, $m_2 = 0.1000 - j1.287$

3. $n_1 = 0.1001$, $n_2 = 0.04994 + j1.822$

4. From experimental results γ was estimated to be 100°. The results of the remaining calculations for this point are

shown in Table 5. The values of β_1 and β_2 were not equal but were fairly close so another value of γ was tried that of 95°. It was known from previous calculations that if $\beta_1 > \beta_2$, the first estimate of γ had been too high, and that was the reason for trying $\gamma = 95°$. The results of the calculation for $\gamma = 95°$ are also shown in Table 5, and now $\beta_1 < \beta_2$, so the crossing point must be between $\gamma = 100°$ and $\gamma = 95°$. If the curves of β_1 and β_2 are assumed as straight lines over the small variation in γ from 95° to 100°, the value of γ at crossing can be calculated as 99°5'.

Similar calculated values of γ were obtained for other pairs of values of a and b, and these are presented in Table 6. Table 6 contains none of the values calculated for the special cases considered in Appendices C and D. The angle β may be calculated once the value of γ has been obtained, by using the scheme of calculation presented in Table 5. Since for this value of γ, $\beta_1 = \beta_2$, either β_1 or β_2 may be taken as the calculated value of β. The angle α may then be obtained by the relation $\alpha = \beta - \gamma$. Calculated values of both β and α are shown in Table 6.

TABLE 5

Computation of γ for a = 0.6, b = 5.0

γ	100°	95°
H_1	-0.6734	-0.8210
H_2	1.924	1.438
H_3	8.090	8.153
D	-131.8	-136.4
K_1	0.5545	0.5539
K_2	-0.2860	-0.1910
K_3	0.3575	0.2727
K_4	-0.1479	-0.09720
K_5	0.01523	-0.005965
K_6	0.4978	0.517
K_7	-0.2436	-0.1868
K_8	0.7209	0.6962
K_9	-0.04958	-0.01766
G_1	-1.051	-0.9047
G_2	1.581	1.753
G_3	-1.075	-1.160
M_1	-2.114	-2.178
M_2	-2.433	-2.491
M_3	0.9217	0.8864
N_1	0.8186	0.7346
N_2	-0.04237	-0.1035
N_3	1.739	1.692
β_1	$123^{\circ}50'$	$118^{\circ}5'$
β_2	$123^{\circ}22'$	$120^{\circ}10'$

TABLE 6

Calculated Results for Various Values of a and b

a	b	γ	β	α	E_{dc}/E_m	r per cent	P_T
5.0	1.0	134°	138°36'	4°36'	0.645	2.046	5.213
2.0	5.0	63°46'	99°56'	36°10'	0.8142	2.163	11.63
2.0	2.0	99°52'	117°27'	17°35'	0.7043	4.337	7.13
1.0	5.0	66°45'	100°43'	33°58'	0.7935	5.73	11.09
1.0	2.0	107°	122°10'	15°10'	0.688	10.82	6.956
1.0	1.0	147°2'	150°30'	3°28'	0.6438	12.58	4.94
0.6	5.0	99°5'	122°47'	23°40'	0.6927	21.42	10.32
0.6	2.0	132°33'	143°39'	11°7'	0.6657	27.00	5.96
0.6	1.0	155°	158°21'	3°2'	0.641	24.69	4.55
any value	∞	0°	90°	90°	1.000	0	∞

E. The Average Output Voltage

The ratio of the average voltage across the load resistor R to the maximum value of the sinusoidal voltage across one half of the transformer secondary E_{dc}/E_m can be calculated by obtaining first the average current I flowing through the load resistor R. This is the same as twice the average current flowing through each tube, and this fact will be used in finding the expression for I. When the equation (B-3) is evaluated by means of the inverse transform, the following expression is obtained for the tube current in the interval $\alpha/\omega \leq t \leq \beta/\omega$:

$$i_T = \frac{E_m}{\omega L}\Bigg\{ a\cos\omega t + g\sin(\omega t - \lambda) + \frac{\epsilon^{-m_1(\omega t-\alpha)}}{m_2 - m_1}\left[\frac{m_2}{m_1^2+1}\left(-m_1\sin\alpha + \cos\alpha\right) + m_2 x_1 - \frac{x_3}{a}\right] + \frac{\epsilon^{-m_2(\omega t-\alpha)}}{m_1 - m_2}\left[\frac{m_1}{m_2^2+1}\left(-m_2\sin\alpha + \cos\alpha\right) + m_1 x_1 - \frac{x_3}{a}\right]\Bigg\} \tag{B-32}$$

where $\frac{\omega L}{g}$ is the impedance magnitude of the part of the filter after the first condenser and λ is the phase angle of this impedance. With the aid of (B-15) the average load current is

$$I = \frac{1}{\pi}\int_{\alpha}^{\beta} i_T \, d(\omega t) = \frac{1}{\pi}\frac{E_m}{L}\left[a(\sin\beta - \sin\alpha) - \frac{a}{b}(\cos\beta - \cos\alpha) + \frac{a}{b}(x_1 - x_2) - (x_3 - x_4)\right]$$

or

$$E_{dc}/E_m = \frac{1}{\pi}\left[(\cos\alpha - \cos\beta) + b(\sin\beta - \sin\alpha) + (x_1 - x_2) - \frac{b}{a}(x_3 - x_4)\right]. \tag{B-33}$$

For a given a and b, when β is calculated, the numbers K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, and K_9 are also obtained. From (B-26), x_1, x_3, and x_4 may be calculated, and x_2 can be obtained from (B-18). With these values found, the ratio E_{dc}/E_m may be calculated from (B-33). Table 6 gives E_{dc}/E_m for various values of a and b.

F. The Per Cent Ripple Voltage

The per cent ripple r is defined as one hundred times the ratio of the effective value of the fundamental ripple voltage across the load resistor R to the average value of the voltage across the same load resistor. The method of obtaining the ripple voltage will be to calculate it across the first condenser using Fourier series coefficients, and then since only one frequency, that of the fundamental ripple voltage, is involved, ordinary circuit analysis may be used to find the ripple voltage across the load resistor. In the derivation of (B-9), the instantaneous voltage e_1 across the first condenser was obtained. In the interval $(\beta/\omega - T) \leq t \leq \alpha/\omega$, this voltage is

$$e_1 = \frac{E_m}{a} \sum_{\nu=1}^{3} \frac{\epsilon^{-n_\nu(\omega t - \alpha)}\left[-\left(\frac{1}{b} - n_\nu\right)x_1 + \frac{1}{a}x_3 + \left(\frac{\frac{1}{b} - n_\nu}{a n_\nu}\right)x_5\right]}{(n_{\nu+1} - n_\nu)(n_{\nu+2} - n_\nu)} \qquad \text{(B-34)}$$

where $n_4 = n_1$, $n_5 = n_2$, and in the interval $\alpha/\omega \leq t \leq \beta/\omega$

$$e_1 = E_m \sin \omega t. \tag{B-35}$$

The Fourier series for e_1 is

$$e_1 = E_{dc} + E_s \sin 2\omega t + E_c \cos 2\omega t + \ldots\ldots \tag{B-36}$$

where

$$\left.\begin{aligned} E_s &= \frac{2}{\pi}\int_{\beta-\pi}^{\beta} e_1 \sin 2\omega t \; d(\omega t) \\ E_c &= \frac{2}{\pi}\int_{\beta-\pi}^{\beta} e_1 \cos 2\omega t \; d(\omega t). \end{aligned}\right\} \tag{B-37}$$

Equations (B-37) may be evaluated by the use of (B-34), (B-35), and (B-16) to obtain

$$\left.\begin{aligned} \frac{E_s}{E_m} &= \frac{2}{\pi}\left\{\frac{1}{a}\left[-D_1 \sin 2\alpha - 2D_2 \cos 2\alpha + D_3 \sin 2\beta + 2D_4 \cos 2\beta\right] \right. \\ &\qquad \left. + \frac{1}{2}\left(\sin\beta - \sin\alpha\right) - \frac{1}{6}\left(\sin 3\beta - \sin 3\alpha\right)\right\} \\ \frac{E_c}{E_m} &= \frac{2}{\pi}\left\{\frac{1}{a}\left[-D_1 \cos 2\alpha + 2D_2 \sin 2\alpha + D_3 \cos 2\beta - 2D_4 \sin 2\beta\right] \right. \\ &\qquad \left. + \frac{1}{2}\left(\cos\beta - \cos\alpha\right) - \frac{1}{6}\left(\cos 3\beta - \cos 3\alpha\right)\right\} \end{aligned}\right\} \tag{B-38}$$

where

$$\left.\begin{aligned} D_0 &= \frac{1}{b^2}\left(\frac{1}{a} - 4\right)^2 + 16\left(\frac{1}{a} - 2\right)^2 \\ D_1 &= \frac{1}{D_0}\left[\kappa_1\left(\frac{4}{b^2} + 16 - \frac{8}{a} - \frac{1}{ab^2}\right) + \frac{\kappa_3}{ab}\left(\frac{1}{a} - 4\right) + \frac{\kappa_5}{a^2 b}\right] \\ D_2 &= \frac{1}{D_0}\left[-\frac{\kappa_1}{ab} + \kappa_3\left(\frac{2}{a^2} - \frac{4}{a}\right) - \frac{\kappa_5}{a}\left(12 - \frac{2}{a} + \frac{1}{b^2} - 4\frac{a}{b} - 16a\right)\right] \\ D_3 &= \frac{1}{D_0}\left[\kappa_2\left(\frac{4}{b^2} + 16 - \frac{8}{a} - \frac{1}{ab^2}\right) + \frac{\kappa_4}{ab}\left(\frac{1}{a} - 4\right) + \frac{\kappa_6}{a^2 b}\right] \\ D_4 &= \frac{1}{D_0}\left[-\frac{\kappa_2}{ab} + \kappa_4\left(\frac{2}{a^2} - \frac{4}{a}\right) - \frac{\kappa_6}{a}\left(12 - \frac{2}{a} + \frac{1}{b^2} - 4\frac{a}{b} - 16a\right)\right]. \end{aligned}\right\} \tag{B-39}$$

The quantities x_5 and x_8 may be calculated from (B-17) and (B-19), D_0, D_1, D_2, D_3, and D_4 may be calculated by means of (B-39), and E_s/E_m and E_c/E_m by use of (B-38). The effective value of the fundamental ripple voltage across the first condenser is

$$\frac{1}{\sqrt{2}}\sqrt{E_s^2 + E_c^2}.$$

This voltage is applied to the rest of the filter circuit composed of the inductor in series with the parallel combination of the second condenser and the load resistor. By ordinary circuit analysis for one frequency it can be shown that the ratio of the output voltage across the load resistor to the input voltage is

$$\frac{\frac{1}{4a}}{\sqrt{\frac{1}{4b^2} + \left(\frac{1}{4a} - 1\right)^2}},$$

and thus the effective value of the fundamental ripple voltage across the load resistor R is

$$\frac{1}{\sqrt{2}}\sqrt{E_s^2 + E_c^2}\;\frac{\frac{1}{4a}}{\sqrt{\frac{1}{4b^2} + \left(\frac{1}{4a} - 1\right)^2}}.$$

The per cent ripple r is

$$r = 100\left(\frac{\frac{1}{\sqrt{2}}\sqrt{\left(\frac{E_s}{E_m}\right)^2 + \left(\frac{E_c}{E_m}\right)^2}}{E_{dc}/E_m}\right)\left(\frac{\frac{1}{4a}}{\sqrt{\frac{1}{4b^2} + \left(\frac{1}{4a} - 1\right)^2}}\right), \quad \text{(B-40)}$$

and it is given for various values of a and b in Table 6.

G. The Peak Tube Current

The ratio of the peak to average tube current P_T is useful in selecting the tube to be used or in selecting the constants of the filter circuit to be used with a given tube. The peak tube current i_m occurs at $t = \alpha/\omega$, and it may be evaluated by putting this value of t in (B-32):

$$i_m = \frac{E_m}{\omega L}\left(a \cos\alpha + x_1\right) .$$

The ratio of peak to average current is

$$P_T = \frac{2b(\cos\alpha + x_1/a)}{E_{dc}/E_m} . \qquad \text{(B-41)}$$

This ratio has been calculated for various values of a and b, and the results are given in Table 6.

XIV. APPENDIX C-THE SPECIAL CASE OF $a \to \infty$

A. The Solution of the Circuit Equations

The special case, $a \to \infty$, implies that the inductance L is infinitely large and that the current i_L is constant and equal to the average load current I. The circuit equations obtained from Fig. 3(a) become

$$\left.\begin{aligned} & i_L = i = I \\ & q_2/C = IR \text{ and } i_2 = 0 \\ & E_m \sin \omega t = q_1/C \\ & \quad i_T = i_1 + I = \omega C E_m \cos \omega t + I. \end{aligned}\right\} \quad \text{(C-1)}$$

The equations (C-1) are valid in the interval $\alpha/\omega \leq t \leq \beta/\omega$. By the use of Fig. 3(b), circuit equations which hold in the interval $\beta/\omega \leq t \leq T + (\alpha/\omega)$ may be obtained. These are

$$\left.\begin{aligned} & i_1 = -i_L = -i = -I \\ & \quad q_2/C = IR \text{ and } i_2 = 0. \end{aligned}\right\} \quad \text{(C-2)}$$

To obtain the equation for the tube conducting angle γ, it is necessary to follow the action of the first condenser during one-half cycle. At $t = \alpha/\omega$, the tube starts conducting, and

$$q_1(\alpha/\omega) = CE_m \sin \alpha. \quad \text{(C-3)}$$

In the interval of conduction, $\alpha/\omega \leq t \leq \beta/\omega$:

$$q_1(t) = CE_m \sin \omega t. \quad \text{(C-4)}$$

At $t = \beta/\omega$, the tube stops conducting and $i_T = i_1 + i_L = 0$.

$$I = -\omega C E_m \cos\beta. \qquad \text{(C-5)}$$

$$q_1(\beta/\omega) = C E_m \sin\beta. \qquad \text{(C-6)}$$

For the interval in which the tube is not conducting

$$q_1(t) = q_1(\beta/\omega) - I(t - \beta/\omega). \qquad \text{(C-7)}$$

At $t = T + (\alpha/\omega)$, the second tube starts conducting, and

$$q_1(\alpha/\omega) = q_1(\beta/\omega) - I(\pi + \alpha - \beta)/\omega. \qquad \text{(C-8)}$$

So far there are five unknowns which occur in the equations, and these are α, β, $q_1(\alpha/\omega)$, $q_1(\beta/\omega)$ and I. There are only four equations involving these unknowns, and these are (C-3), (C-5), (C-6), and (C-8). The fifth equation may be obtained by recognizing that in the steady state the average voltage across the inductor must be zero, and from this the average voltage RI across the load resistor must be equal to the average voltage of the first condenser. Expressed mathematically

$$RI = \frac{1}{\pi}\int_{\alpha}^{\alpha+\pi} (q_1/C)\, d(\omega t),$$

and this may be evaluated with the aid of (C-4) and (C-7) to obtain

$$\pi RI = E_m(\cos\alpha - \cos\beta) + (\pi - \gamma)\left[E_m \sin\beta - \frac{I}{2\omega C}(\pi - \gamma)\right]. \qquad \text{(C-9)}$$

The five equations may be solved in the following manner. From (C-5)

$$E_{dc}/E_m = -b\cos\beta. \qquad \text{(C-10)}$$

From (C-3), (C-6) and (C-8)

$$E_{dc}/E_m = \frac{b(\sin\beta - \sin\alpha)}{\pi - \gamma} \ . \qquad \text{(C-11)}$$

From (C-9)

$$E_{dc}/E_m = \frac{\cos\alpha - \cos\beta + \frac{(\pi-\gamma)}{2}\sin\beta}{\pi + \frac{(\pi-\gamma)^2}{2b}} \ . \qquad \text{(C-12)}$$

The three equations (C-10), (C-11), and (C-12) become by the use of $\alpha = \beta - \gamma$ and by the elimination of E_{dc}/E_m, the following equations

$$\tan\beta = -\frac{(\pi-\gamma)+\sin\gamma}{1-\cos\gamma} = -\frac{b\pi + (1/2)(\pi-\gamma)^2 - (1-\cos\gamma)}{\pi-\gamma+\sin\gamma} . \qquad \text{(C-13)}$$

The part of (C-13) involving γ may be manipulated into the form

$$b = \frac{1}{2\pi}\left[(\pi-\gamma)\left(\frac{\sin\gamma}{1-\cos\gamma}\right)+2\right]^2 \qquad \text{(C-14)}$$

which gives γ as an implicit function of b.

B. The Calculation of the Characteristics

The first characteristic curve desired for this special case is that of γ versus b. Such a curve can be obtained from (C-14) by assuming various values of γ and calculating the corresponding values of b. The results of this calculation are shown in Table 7. For values of $b < 2/\pi = 0.6366$, the analysis above does not hold, and this case (the non-cut-off case) will be discussed further on in this appendix. The angle β may now be calculated from (C-13), and the angle α from $\alpha = \beta - \gamma$. These are also presented in Table 7.

TABLE 7

Calculated Values for the Special Case $a \rightarrow \infty$

b	γ	β	α	E_{dc}/E_m	P_T
∞	0°00'	90°00'	90°00'	1.000	∞
79.6	16°00'	90°42'	74°42'	0.972	45.23
22.05	30°00'	92°28'	62°28'	0.948	23.47
9.38	45°00'	95°28'	50°28'	0.893	15.38
5.04	60°00'	99°35'	39°35'	0.839	11.26
2.088	90°00'	111°15'	21°15'	0.735	7.142
1.000	124°36'	131°12'	6°36'	0.659	5.016
0.862	135°00'	138°49'	3°49'	0.648	4.655
0.6366	180°00'	180°00'	0°00'	0.6366	4.000

The ratio of the average voltage across the load resistor R to the maximum value of the sinusoidal voltage across half of the transformer secondary E_{dc}/E_m may be calculated from (C-10) and is presented in Table 7. The per cent ripple r across the load resistor R is defined as the ratio of the effective fundamental ripple voltage across the load resistor to the average voltage E_{dc} across the same load resistor.

For this special case the per cent ripple r across the load resistor is zero for all values of b. The peak tube current is

$$i_m = \omega C E_m \cos\alpha + I,$$

and the ratio of peak to average is

$$P_T = 2i_m/I = \frac{2b\cos\alpha}{E_{dc}/E_m} + 2. \qquad \text{(C-15)}$$

The ratio has also been calculated and is given in Table 7.

For $0 \leq b \leq 2/\pi = 0.6366$, each tube conducts 180°, and this is called the non-cut-off case because the circuit no longer stops current conduction or cuts-off the flow of current in the tubes. For all values of b in this range, $\gamma = 180^\circ$, $\beta = 180^\circ$, $\alpha = 0^\circ$, $E_{dc}/E_m = 2/\pi = 0.6366$, and $r = 0$. Since the instantaneous current in the tube is

$$i_T = C E_m \cos\omega t + I,$$

the ratio of peak to average tube current is

$$P_T = b\pi + 2 \qquad \text{(C-16)}$$

which ranges between 4 for $b = 0.6366$ to 2 for $b = 0$.

XV. APPENDIX D-THE NON-CUT-OFF CASE

A. The Calculation of the Boundary Values

The first part of this appendix will be devoted to the finding of the boundary between the cut-off and non-cut-off cases. The tube current i_T has its minimum point at $t = T$, and on the boundary the tube current should just reach zero. Putting $i_T = 0$ at $t = T$ in (B-32) gives:

$$-a + (1+2U_2)\, g \sin\lambda - 2U_1 g \cos\lambda = 0 \qquad \text{(D-1)}$$

where

$$\left.\begin{aligned} U_1 &= \frac{\frac{1}{1-\epsilon^{m_1\pi}} - \frac{1}{1-\epsilon^{m_2\pi}}}{m_1 - m_2} = \frac{\frac{1}{m_{1C}} \sin m_{1C}\pi}{\epsilon^{-m_{1R}\pi} - 2\cos m_{1C}\pi + \epsilon^{m_{1R}\pi}} \\ U_2 &= \frac{m_2\left(\frac{1}{1-\epsilon^{m_1\pi}}\right) - m_1\left(\frac{1}{1-\epsilon^{m_2\pi}}\right)}{m_1 - m_2} = \frac{-\epsilon^{-m_{1R}\pi} + \frac{m_{1R}}{m_{1C}} \sin m_{1C}\pi + \cos m_{1C}\pi}{\epsilon^{-m_{1R}\pi} - 2\cos m_{1C}\pi + \epsilon^{m_{1R}\pi}}. \end{aligned}\right\} \qquad \text{(D-2)}$$

For a given value of a, (D-1) may be solved for the corresponding value of b which gives a point on the boundary between cut-off and non-cut-off. The equation (D-1) cannot be solved explicitly for b, and a graphical method of solution was used. This method is given in outline as follows:

1. For a given a, estimate the value of b. Since for the $a \rightarrow \infty$ case, on the boundary $b = 2/\pi$, this value of b was usually the first estimated value.

2. Calculate m_1 and m_2.

3. Calculate U_1 and U_2 from (D-2).

4. Calculate [see (D-1)]

$$y = -a + (1 + 2U_2)\, g \sin\lambda - 2U_1\, g \cos\lambda \,. \qquad \text{(D-3)}$$

5. For other estimated values of b, repeat parts 2, 3, and 4.

6. Plot y versus b, and where y becomes zero is the required value of b.

A number of the pair of values of a and b on this boundary have been calculated and are given in Table 8. If for a given a, the chosen b is greater than the values given in Table 8, the tubes conduct less than 180°, and the cut-off

TABLE 8

Points on the Boundary Line Between the Cut-Off and Non-Cut-Off Cases

a	b	P_T
∞	0.6366	4.000
10.0	0.6388	4.015
5.0	0.6410	4.027
3.5	0.6380	4.009
2.0	0.6338	3.983
1.0	0.6140	3.859
0.6	0.5700	3.582

case applies. If on the other hand, the b is less than the values in the table, the tubes conduct through 180 degrees, and the non-cut-off case should be used.

B. The Calculation of the Characteristics

The tube angles are constant for the non-cut-off case and have the following values,

$$\gamma = 180^\circ$$
$$\beta = 180^\circ \qquad \text{(D-4)}$$
$$\alpha = 0^\circ$$

The ratio E_{dc}/E_m is constant and is equal to $2/\pi = 0.6366$. The ripple r varies with a and b and may be calculated by the use of (B-38), (B-39), and (B-40) to obtain

$$r = \frac{47.14}{4a\sqrt{\frac{1}{4b^2} + \left(\frac{1}{4a} - 1\right)^2}} \qquad \text{(D-5)}$$

Since $x_1 = x_2$, $x_3 = x_4$, and $x_5 = x_6$ for this case, equations (B-15) give

$$\left.\begin{aligned} m_2x_1-(x_3/a) &= \frac{m_2}{m_1^2+1}\left(\frac{1+\epsilon^{-m_1\pi}}{1-\epsilon^{-m_1\pi}}\right) \\ m_1x_1-(x_3/a) &= \frac{m_1}{m_2^2+1}\left(\frac{1+\epsilon^{-m_2\pi}}{1-\epsilon^{-m_2\pi}}\right). \end{aligned}\right\} \text{(D-6)}$$

The tube current may be obtained by substituting (D-6) into (B-32):

$$i_T = \frac{E_m}{\omega L}\left\{a\cos\omega t + q\sin(\omega t-\lambda)\right.$$

$$\left.-\frac{2}{m_1-m_2}\left[\frac{m_2\epsilon^{-m_1\omega t}}{(m_1^2+1)(1-\epsilon^{-m_1\pi})}-\frac{m_1\epsilon^{-m_2\omega t}}{(m_2^2+1)(1-\epsilon^{-m_2\pi})}\right]\right\}. \quad \text{(D-7)}$$

The maximum tube current i_m occurs at $t = 0$, and the ratio of peak to average tube current is $P_T = 2i_m/I$, or from (D-7)

$$P_T = \frac{\pi b}{a}\left\{a - q\sin\lambda - \frac{2}{m_1-m_2}\left[\frac{m_2}{(m_1^2+1)(1-\epsilon^{-m_1\pi})}-\frac{m_1}{(m_2^2+1)(1-\epsilon^{-m_2\pi})}\right]\right\}$$

$$= \frac{\pi b}{a}\left[a + (1+2U_2)q\sin\lambda - 2U_1 q\cos\lambda\right]. \quad \text{(D-8)}$$

On the boundary between the cut-off and non-cut-off cases (D-8) may be simplified by the use of (D-3):

$$P_T = 2\pi b. \quad \text{(D-9)}$$

Equation (D-9) checks with the result obtained from (B-41), and the values of P_T on the boundary calculated from (D-9) are given in Table 8. When $b = 0$ (short-circuit), $P_T = 2.0$, as given by (D-8).